# 2 Functions and equations

**Assessment statements**

2.1 Concept of function $f: x \mapsto f(x)$; domain, range, image (value).
Composite functions $(f \circ g)$; identity function. Inverse function $f^{-1}$.

2.2 The graph of a function; its equation $y = f(x)$.
Function graphing skills: use of a GDC to graph a variety of functions; investigation of key features of graphs.
Solutions of equations graphically.

2.3 Transformations of graphs: translations; stretches; reflections in the axes.
The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph $y = f(x)$.

2.4 The reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$: its graph; its self-inverse nature.

2.5 The quadratic function $x \rightarrow ax^2 + bx + c$: its graph, $y$-intercept $(0, c)$, axis of symmetry $x = -\frac{b}{2a}$.
The form $x \rightarrow a(x - h)^2 + k$: vertex $(h, k)$.
The form $x \rightarrow a(x - p)(x - q)$: $x$-intercepts $(p, 0)$ and $(q, 0)$.

2.6 The solution of $ax^2 + bx + c = 0$, $a \neq 0$.
The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.

## Introduction

This chapter looks at functions and considers how they can be used in describing physical phenomena. We also investigate composite and inverse functions, and transformations such as translations, stretches and reflections. Quadratic functions are treated graphically and algebraically.

## 2.1 Relations and functions

### Relations

There are different scales for measuring temperature. Two of the more commonly used are the Celsius scale and the Fahrenheit scale. A temperature recorded in one scale can be converted to a value in the other scale, based on the fact that there is a constant relationship between the two sets of numbers in each scale. If the variable $C$ represents degrees Celsius and the variable $F$ represents degrees Fahrenheit, this relationship can be expressed by the following equation that converts Celsius to Fahrenheit: $F = \frac{9}{5}C + 32$.

Most countries except the United States use the Celsius scale, invented by the Swedish scientist Anders Celsius (1701-1744). The United States uses the earlier Fahrenheit scale, invented by the Dutch scientist Gabriel Daniel Fahrenheit (1686-1736). A citizen of the USA travelling to other parts of the world will need to convert from degrees Celsius to degrees Fahrenheit.

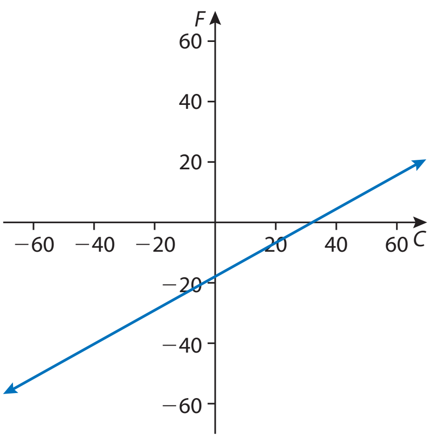
Many mathematical relationships concern how two sets of numbers relate to one another – and often the best way to express this is with an algebraic equation in two variables. If it's not too difficult, we find it useful to express one variable in terms of the other. For example, in the previous equation, $F$ is written in terms of $C$ – making $C$ the **independent variable** and $F$ the **dependent variable**. Since $F$ is written in terms of $C$, it is easiest for you to first substitute in a value for $C$, and then evaluate the expression to determine the value of $F$. In other words, the value of $F$ is *dependent* upon the value of $C$ that is chosen *independent* of $F$.

A **relation** is a rule that determines how a value of the independent variable corresponds – or is **mapped** – to a value of the dependent variable. A temperature of 30 degrees Celsius corresponds to 86 degrees Fahrenheit.

$$F = \tfrac{9}{5}(30) + 32 = 54 + 32 = 86$$

Along with equations, other useful ways of representing a relation include a gr aph of the equation on a **Cartesian coordinate system** (also called a **rectangular coordinate system**), a **table**, a set of **ordered pairs**, or a **mapping**. These are illustrated below for the equation $F = \tfrac{9}{5}C + 32$.

### Graph



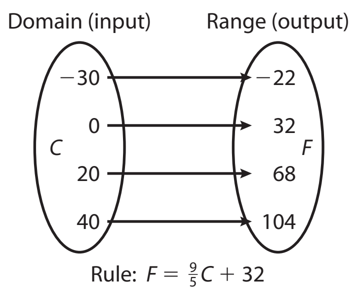

### Table

| Celsius (C) | Fahrenheit (F) |
|---|---|
| −40 | −40 |
| −30 | −22 |
| −20 | −4 |
| −10 | 14 |
| 0 | 32 |
| 10 | 50 |
| 20 | 68 |
| 30 | 86 |
| 40 | 104 |

### Ordered pairs

The graph of the equation $F = \tfrac{9}{5}C + 32$ is a line consisting of an infinite set of ordered pairs $(C, F)$ – each is a solution of the equation. The following set includes some of the ordered pairs on the line: $\{(-30, -22), (0, 32), (20, 68), (40, 104)\}$.

### Mapping




The largest possible set of values for the independent variable (the **input** set) is called the **domain** – and the set of resulting values for the dependent variable (the **output** set) is called the **range**. In the context of a mapping, each value in the domain is mapped to its **image** in the range.

**René Descartes**

The Cartesian coordinate system is named in honour of the French mathematician and philosopher René Descartes (1596-1650). Descartes stimulated a revolution in the study of mathematics by merging its two major fields – algebra and geometry. With his coordinate system utilizing ordered pairs (*Cartesian coordinates*) of real numbers, geometric concepts could be formulated analytically and algebraic concepts (e.g. relationships between two variables) could be viewed graphically. Descartes initiated something that is very helpful to all students of mathematics – that is, considering mathematical concepts from multiple perspectives: graphical (visual) and analytical (algebraic).

• **Hint:** The coordinate system for the graph of an equation has the independent variable on the horizontal axis and the dependent variable on the vertical axis.

## Functions

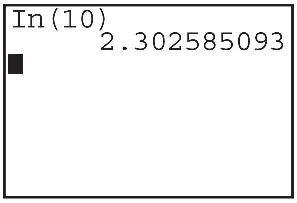
If the relation is such that each number (or **element**) in the domain produces one and only one number in the range, the relation is called a **function**. Common sense tells us that each numerical temperature in degrees Celsius ($C$) will convert (or correspond) to only one temperature in degrees Fahrenheit ($F$). Therefore, the relation given by the equation $F = \frac{9}{5}C + 32$ is a function - any chosen value of $C$ corresponds to exactly one value of $F$. The idea that a function is a rule that assigns to each number in the domain a unique number in the range is formally defined below.

**Definition of a function**

A **function** is a correspondence (**mapping**) between two sets X and Y in which each element of set X corresponds to (maps to) exactly one element of set Y. The **domain** is set X (**independent variable**) and the **range** is set Y (**dependent variable**).

Not only are functions important in the study of mathematics and science, we encounter and use them routinely – often in the form of tables. Examples include height and weight charts, income tax tables, loan payment schedules, and time and temperature charts. The importance of functions in mathematics is evident from the many functions that are installed on your GDC.

For example, the keys labelled SIN $x^{-1}$ LN $\sqrt{\ }$ each represent a function, because for each input (entry) there is only one output (answer). The calculator screen image shows that for the function $y = \ln x$, the input of $x = 10$ has only one output of $y \approx 2.302\,585\,093$.



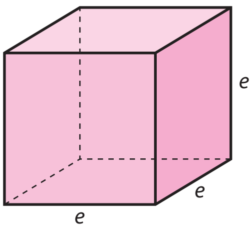

For many physical phenomena, we observe that one quantity depends on another. For example, the boiling point of water depends on elevation above sea level; the time for a pendulum to swing through one cycle (its period) depends on the length of the pendulum; and the area of a circle depends on its radius. The word **function** is used to describe this dependence of one quantity on another – i.e. how the value of an independent variable determines the value of a dependent variable.

- Boiling point is a function of elevation (elevation determines boiling point).
- The period of a pendulum is a function of its length (length determines period).
- The area of a circle is a function of its radius (radius determines area).

### Example 1

a) Express the volume $V$ of a cube as a function of the length $e$ of each edge.

b) Express the volume $V$ of a cube as a function of its surface area $S$.



### Solution

a) $V$ as a function of $e$ is $V = e^3$.

b) The surface area of the cube consists of six squares each with an area of $e^2$. Hence, the surface area is $6e^2$; that is, $S = 6e^2$. We need to write $V$ in terms of $S$. We can do this by first expressing $e$ in terms of $S$, and then substituting this expression in for $e$ in the equation $V = e^3$.

$S = 6e^2 \Rightarrow e^2 = \frac{S}{6} \Rightarrow e = \sqrt{\frac{S}{6}}$.

Substituting,

$$V = \left(\sqrt{\frac{S}{6}}\right)^3 = \frac{(S^{\frac{1}{2}})^3}{(6^{\frac{1}{2}})^3} = \frac{S^{\frac{3}{2}}}{6^{\frac{3}{2}}} = \frac{S^1 \cdot S^{\frac{1}{2}}}{6^1 \cdot 6^{\frac{1}{2}}} = \frac{S}{6}\sqrt{\frac{S}{6}}.$$

$V$ as a function of $S$ is $V = \frac{S}{6}\sqrt{\frac{S}{6}}$.

## Domain and range of a function

The domain of a function may be stated explicitly, or it may be implied by the expression that defines the function. If not explicitly stated, the domain of a function is the set of all real numbers for which the expression is defined as a real number. For example, if a certain value of $x$ is substituted into the algebraic expression defining a function and it causes division by zero or the square root of a negative number (both undefined in the real numbers) to occur, that value of $x$ cannot be in the domain. The domain of a function may also be implied by the physical context or limitations that exist. Usually the range of a function is not given explicitly and is determined by analyzing the output of the function for all values of the input. The range of a function is often more difficult to find than the domain, and analyzing the graph of a function is very helpful in determining it. A combination of algebraic and graphical analysis is very useful in determining the domain and range of a function.

### Example 2

Find the domain of each of the following functions.

a) $\{(-6, -3), (-1, 0), (2, 3), (3, 0), (5, 4)\}$

b) Area of a circle: $A = \pi r^2$

c) $y = \frac{1}{x}$

d) $y = \sqrt{x}$

### Solution

a) The function consists of a set of ordered pairs. The domain of the function consists of all first coordinates of the ordered pairs. Therefore, the domain is the set $\{-6, -1, 2, 3, 5\}$.

b) The physical context tells you that a circle cannot have a negative radius. You can only choose values for the radius ($r$) that are greater than or equal to zero. Therefore, the domain is the set of all real numbers such that $r \geqslant 0$.

c) The value of $x = 0$ cannot be included in the domain because division by zero is not defined for real numbers. Therefore, the domain is the set of all real numbers except zero ($x \neq 0$).

d) Any negative values of $x$ cannot be in the domain because the square root of a negative number is not a real number. Therefore, the domain is all real numbers such that $x \geqslant 0$.

## Determining if a relation is a function

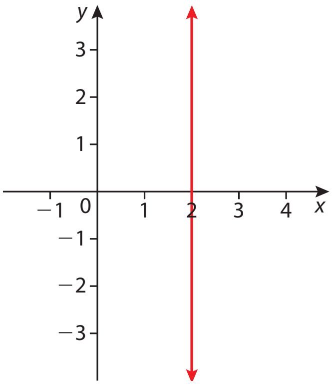


**Figure 2.1**

Some relations are not functions – and because of the mathematical significance of functions it is important for us to be able to determine when a relation is, or is not, a function. It follows from the definition of a function that a relation for which a value of the domain ($x$) corresponds to (or determines) *more than one* value in the range ($y$) is *not* a function. Any two points (ordered pairs $(x, y)$) on a *vertical* line have the same $x$-coordinate. Although a trivial case, it is useful to recognize that the equation for a vertical line, $x = 2$ for example (see Figure 2.1), is a relation but *not* a function. The points with coordinates $(2, -3)$, $(2, 0)$ and $(2, 4)$ are all solutions to the equation $x = 2$. The number two is the only element in the domain of $x = 2$ but it is mapped to *more than one* value in the range ($-3$, 0 and 4, for example). It follows that if a vertical line intersects the graph of a relation at more than one point, then a value in the domain ($x$) corresponds to more than one value in the range ($y$) and, hence, the relation is *not* a function. This argument provides an alternative definition of a function and also a convenient visual test to determine whether or not the graph of a relation represents a function.

**Alternative definition of a function**

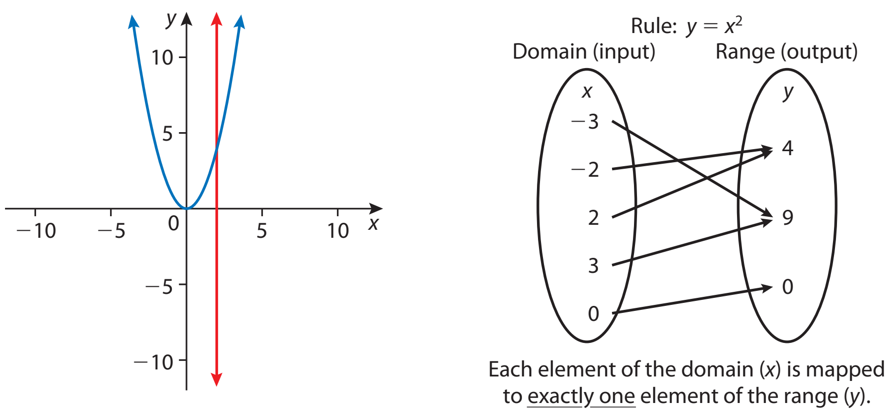
A **function** is a relation in which no two different ordered pairs have the same first coordinate.

**Vertical line test for functions**

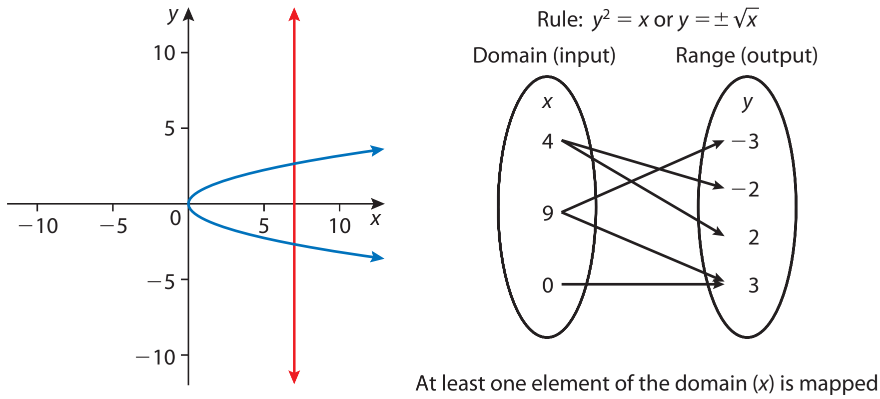
A vertical line intersects the graph of a function at no more than one point.

As the graph in Figure 2.2 clearly shows, a vertical line will intersect the graph of $y = x^2$ at no more than one point – therefore, the relation $y = x^2$ is a function.

**Figure 2.2**



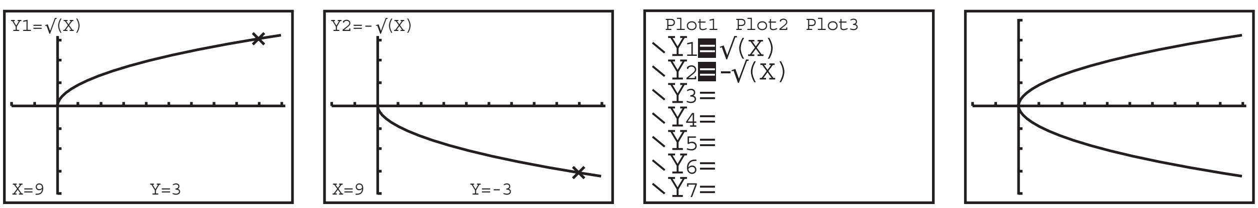
In contrast, the graph of the equation $y^2 = x$ is a 'sideways' parabola that can clearly be intersected more than once by a vertical line (see Figure 2.3). There are at least two ordered pairs having the same $x$-coordinate but different $y$-coordinates (for example, (9, 3) and (9, −3)). Therefore, the relation $y^2 = x$ fails the vertical line test indicating that it does *not* represent a function.



At least one element of the domain ($x$) is mapped to more than one element of the range ($y$).

**Figure 2.3**

● **Hint:** Note that to graph the equation $y^2 = x$ on your GDC, you need to solve for $y$ in terms of $x$. The result is two separate equations: $y = \sqrt{x}$ and $y = -\sqrt{x}$, (or $y = \pm\sqrt{x}$). Each is one-half of the 'sideways' parabola. Although each represents a function (vertical line test), the combination of the two is a complete graph of $y^2 = x$ that clearly does not satisfy either definition of a function.




## Example 3

What is the domain and range for the function $y = x^2$?

### *Solution*

- *Algebraic analysis*: Squaring any real number produces another real number. Therefore, the domain of $y = x^2$ is the set of all real numbers ($\mathbb{R}$). What about the range? Since the square of any positive or negative number will be positive and the square of zero is zero, the range is the set of all real numbers greater than or equal to zero.
- *Graphical analysis*: For the domain, focus on the $x$-axis and *horizontally* scan the graph from $-\infty$ to $+\infty$. There are no 'gaps' or blank regions in the graph and the parabola will continue to get 'wider' as $x$ goes to either $-\infty$ or $+\infty$. Therefore, the domain is all real numbers. For the range, focus on the $y$-axis and *verticaly* scan from $-\infty$ or $+\infty$. The parabola will continue 'higher' as $y$ goes to $+\infty$, but the graph does not go below the $x$-axis. The parabola has no points with negative $y$-coordinates. Therefore, the range is the set of real numbers greater than or equal to zero. See Figure 2.4.

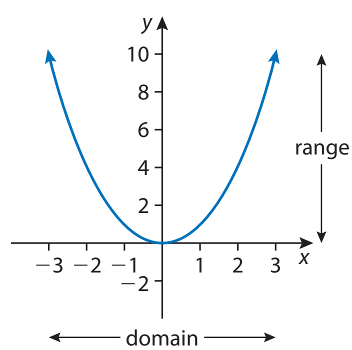



**Figure 2.4**

**Table 2.1** Different ways of expressing the domain and range of $y = x^2$.

| Description in words | Interval notation (both formats) |
|---|---|
| domain is any real number | domain is $\{x : x \in \mathbb{R}\}$ or domain is $x \in ]-\infty, \infty[$ |
| range is any real number greater than or equal to zero | range is $\{y : y \geqslant 0\}$ or range is $y \in [0, \infty[$ |

• **Hint:** The infinity symbol $\infty$ does *not* represent a number. When $\infty$ or $-\infty$ is used in interval notation, it is being used as a convenient notational device to indicate that an interval has no endpoint in a certain direction.

• **Hint:** When asked to determine the domain and range of a function, it is wise for you to conduct both algebraic and graphical analysis – and not rely too much on either approach. For graphical analysis of a function, producing a *comprehensive graph* on your GDC is essential – and an essential skill for this course.

## Function notation

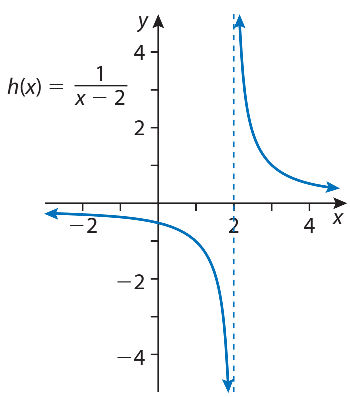
It is common practice to assign a name to a function – usually a single letter with $f$, $g$ and $h$ being the most common. Given that the domain (independent) variable is $x$ and the range (dependent) variable is $y$, the symbol $f(x)$, read '$f$ of $x$', denotes the unique value of $y$ that is generated by the value of $x$. This **function notation** was devised by the famous Swiss mathematician Leonhard Euler (1707-1783). Another notation – sometimes referred to as **mapping notation** – is based on the idea that the function $f$ is the rule that maps $x$ to $f(x)$ and is written $f : x \mapsto f(x)$. For each value of $x$ in the domain, the corresponding unique value of $y$ in the range is called the **function value** at $x$, or the **image** of $x$ under $f$. The image of $x$ may be written as $f(x)$ or as $y$. For example, for the function $f(x) = x^2$: '$f(3) = 9$'; or, 'if $x = 3$ then $y = 9$'.

**Table 2.2** Function notation.

| Notation | Description in words |
|---|---|
| $f(x) = x^2$ | 'the function $f$, in terms of $x$, is $x^2$'; or, simply '$f$ of x is $x^2$' |
| $f : x \mapsto x^2$ | 'the function $f$ maps $x$ to $x^2$' |
| $f(3) = 9$ | 'the value of the function $f$ when $x = 3$ is 9'; or, simply '$f$ of 3 equals 9' |
| $f : 3 \mapsto 9$ | 'the image of 3 under the function $f$ is 9' |

### Example 4

Find the domain and range of the function $h : x \mapsto \frac{1}{x-2}$.




***Solution***

- *Algebraic analysis*: The function produces a real number for all $x$, except for $x = 2$ when division by zero occurs. Hence, $x = 2$ is the only real number not in the domain. Since the numerator of $\frac{1}{x-2}$ can never be zero, the value of $y$ cannot be zero. Hence, $y = 0$ is the only real number not in the range.
- *Graphical analysis*: A horizontal scan shows a 'gap' at $x = 2$ dividing the graph of the equation into two branches that both continue indefinitely with no other 'gaps' as $x \rightarrow \pm\infty$. Both branches are **asymptotic** (approach but do not intersect) to the vertical line $x = 2$. This line is a **vertical asymptote** and is drawn as a dashed line (it is *not* part of the graph of the equation). A vertical scan reveals a 'gap' at $y = 0$ ($x$-axis)

with both branches of the graph continuing indefinitely with no other 'gaps' as $y \to \pm\infty$. Both branches are also asymptotic to the $x$-axis. The $x$-axis is a **horizontal asymptote**.

Both approaches confirm the following for $h: x \mapsto \frac{1}{x-2}$:

The domain is $\{x : x \in \mathbb{R}, x \neq 2\}$ or $x \in ]-\infty, 2[ \cup ]2, \infty[$

The range is $\{y : y \in \mathbb{R}, y \neq 0\}$ or $y \in ]-\infty, 0[ \cup ]0, \infty[$

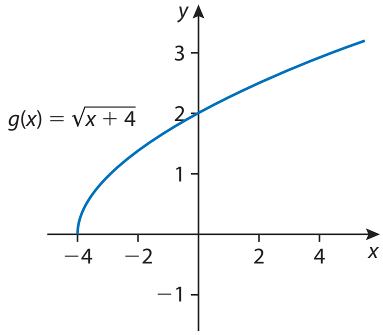
## Example 5

Consider the function $g(x) = \sqrt{x + 4}$.

a) Find: (i) $g(7)$
(ii) $g(32)$
(iii) $g(-4)$

b) Find the values of $x$ for which $g$ is undefined.

c) State the domain and range of $g$.




### Solution

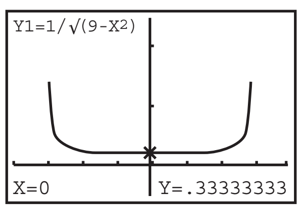
a) (i) $g(7) = \sqrt{7 + 4} = \sqrt{11} \approx 3.32$ (3 significant figures)
(ii) $g(32) = \sqrt{32 + 4} = \sqrt{36} = 6$
(iii) $g(-4) = \sqrt{24 + 4} = \sqrt{0} = 0$

b) $g(x)$ will be undefined (square root of a negative) when $x + 4 < 0$.
$x + 4 < 0 \Rightarrow x < -4$. Therefore, $g(x)$ is undefined when $x < -4$.

c) It follows from the result in b) that the domain of $g$ is $\{x : x \geqslant -4\}$.
The symbol $\sqrt{\phantom{x}}$ stands for the **principal square root** that, by definition, can only give a result that is positive or zero. Therefore, the range of $g$ is $\{y : y \geqslant 0\}$. The domain and range are confirmed by analyzing the graph of the function.

## Example 6

Find the domain and range of the function $f(x) = \frac{1}{\sqrt{9 - x^2}}$.

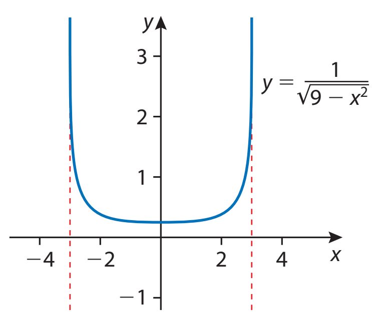


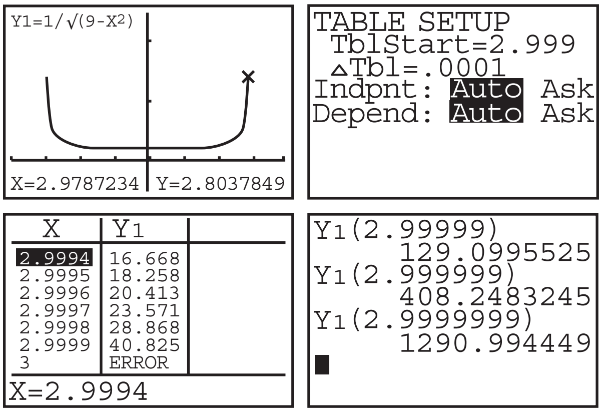

### Solution

The graph of $y = \frac{1}{\sqrt{9 - x^2}}$ on a GDC, shown right, agrees with algebraic analysis indicating that the expression $\frac{1}{\sqrt{9 - x^2}}$ will be positive for all $x$, and is defined only for $-3 < x < 3$.

Further analysis and tracing the graph reveals that $f(x)$ has a minimum at $\left(0, \frac{1}{3}\right)$. The graph on the GDC is misleading in that it appears to show that the function has a maximum value of approximately $y \approx 2.803\,7849$. Can this be correct? A lack of algebraic thinking and over-reliance on your GDC could easily lead to a mistake. The graph abruptly stops its curve upwards because of low screen resolution. Function values should get quite

• **Hint:** As Example 6 illustrates, it is dangerous to completely trust graphs produced on a GDC without also doing some algebraic thinking. It is important to mentally check that the graph shown is comprehensive (shows all important features of the graph), and that the graph agrees with algebraic analysis of the function – e.g. where should the function be zero, positive, negative, undefined, increasing/decreasing without bound, etc.






large for values of $x$ a little less than 3, because the value of $\sqrt{9-x^2}$ will be small making the fraction $\frac{1}{\sqrt{9-x^2}}$ large. Using your GDC to make a table for $f(x)$, or evaluating the function for values of $x$ very close to $-3$ or 3, confirms that as $x$ approaches $-3$ or 3, $y$ increases without bound, i.e. $y$ goes to $+\infty$. Hence, $f(x)$ has vertical asymptotes of $x = -3$ and $x = 3$. This combination of graphical and algebraic analysis leads to the conclusion that the domain of $f(x)$ is $\{x: -3 < x < 3\}$, and the range of $f(x)$ is $\{y: y \geqslant \frac{1}{3}\}$.

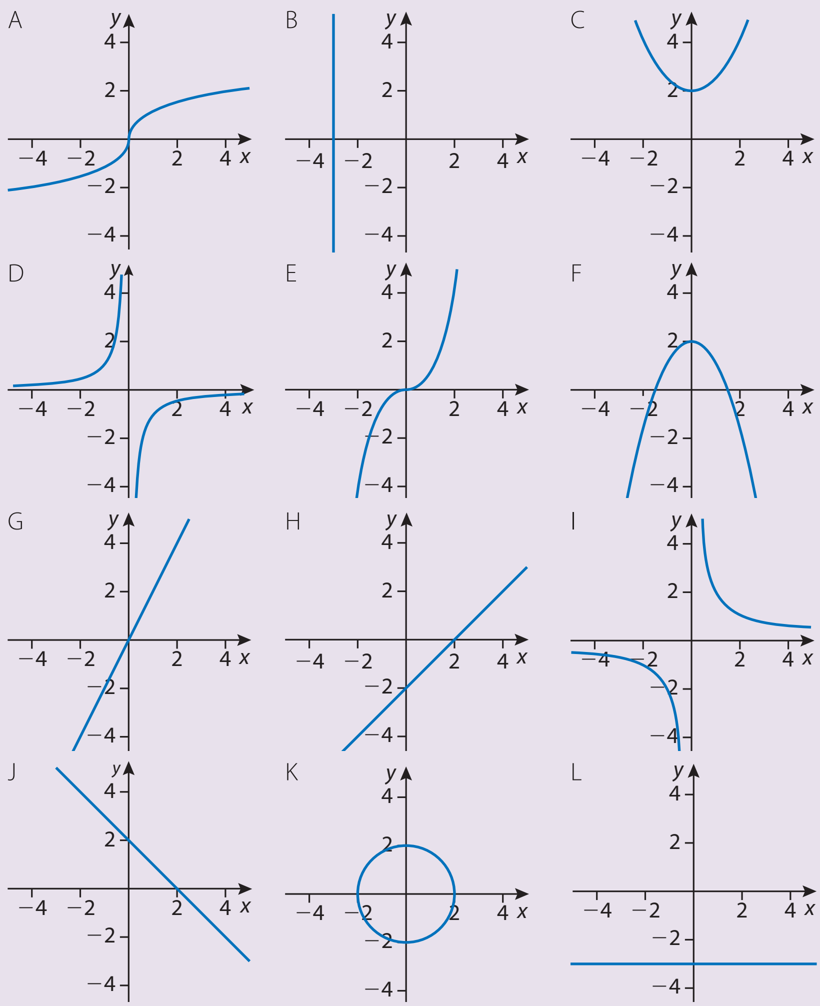
## Exercise 2.1

For each equation 1–9, a) match it with its graph (choices are labelled A to L), and b) state whether or not the equation represents a function – with a justification. Assume that $x$ is the independent variable and $y$ is the dependent variable.

**1** $y = 2x$ **2** $y = -3$ **3** $x - y = 2$

**4** $x^2 + y^2 = 4$ **5** $y = 2 - x$ **6** $y = x^2 + 2$

**7** $y^3 = x$ **8** $y = \frac{2}{x}$ **9** $x^2 + y = 2$



**10** Express the area, $A$, of a circle as a function of its circumference, $C$.

**11** Express the area, $A$, of an equilateral triangle as a function of the length, $\ell$, of each of its sides.

In questions 12–17, find the domain of the function.

**12** $f(x) = \frac{2}{5}x - 7$

**13** $h(x) = x^2 - 4$

**14** $g(t) = \sqrt{3 - t}$

**15** $h(t) = \sqrt[3]{t}$

**16** Volume of a sphere: $V = \frac{4}{3}\pi r^3$

**17** $g(k) = \frac{6}{K^2 - 9}$

**18** Do all linear equations represent a function? Explain.

**19** Find the domain and range of the function $f$ defined as $f: x \mapsto \frac{1}{x - 5}$.

**20** Consider the function $h(x) = \sqrt{x - 4}$.

a) Find: (i) $h(21)$ (ii) $h(53)$ (iii) $h(4)$

b) Find the values of $x$ for which $h$ is undefined.

c) State the domain and range of $h$.

d) Sketch a comprehensive graph of the function.

**21** Find the domain and range of the function $f$ defined as $f(x) = \frac{1}{\sqrt{x^2 - 9}}$ and sketch a comprehensive graph of the function clearly indicating any intercepts or asymptotes.

# 2.2 Composite of functions

## Composite functions

Consider the function in Example 5 in the previous section, $f(x) = \sqrt{x + 4}$. When you evaluate $f(x)$ for a certain value of $x$ in the domain (for example, $x = 5$) it is necessary for you to perform computations in two separate steps in a certain order.

$f(s) = \sqrt{5 + 4} \Rightarrow f(5) = \sqrt{9}$ Step 1: compute the sum of 5 + 4

$\Rightarrow f(5) = 3$ Step 2: compute the square root of 9

Given that the function has two separate evaluation 'steps', $f(x)$ can be seen as a combination of two 'simpler' functions that are performed in a specified order. According to how $f(x)$ is evaluated (as shown above), the simpler function to be performed first is the rule of 'adding 4' and the second is the rule of 'taking the square root.' If $h(x) = x + 4$ and $g(x) = \sqrt{x}$, we can create (compose) the function $f(x)$ from a combination of $h(x)$ and $g(x)$ as follows:

$f(x) = g(h(x))$

$= g(x + 4)$ Step 1: substitute $x + 4$ for $h(x)$ making $x + 4$ the argument of $g(x)$

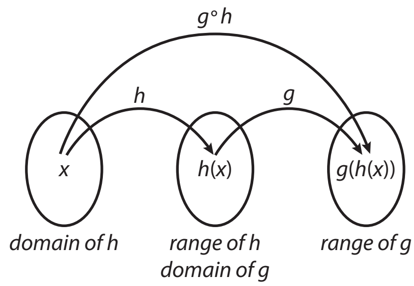
$= \sqrt{x + 4}$ Step 2: apply the function $g(x)$ on the argument $x + 4$

We obtain the rule $\sqrt{x + 4}$ by first applying the rule $x + 4$ and then applying the rule $\sqrt{x}$. A function that is obtained from 'simpler' functions by applying one after another in this way is called a **composite function**. In the example above, $f(x) = \sqrt{x + 4}$ is the **composition** of $h(x) = x + 4$

From the explanation on how $f$ is the composition (or composite) of $g$ and $h$, you can see why a composite function is sometimes referred to as a 'function of a function'. Also note that in the notation $g(h(x))$, the function $h$ that is applied first is written 'inside', and the function $g$ that is applied second is written 'outside'.

followed by $g(x) = \sqrt{x}$. In other words, $f$ is obtained by substituting $h$ into $g$, and can be denoted in function notation by $g(h(x))$ – read '$g$ of $h$ of $x$'.

**Figure 2.5**




We start with a number $x$ in the domain of $h$ and find its image $h(x)$. If this number $h(x)$ is in the domain of $g$, we then compute the value of $g(h(x))$. The resulting composite function is denoted as $(g \circ h)(x)$. See mapping illustration in Figure 2.5.

**Definition of the composition of two functions**

The composition of two functions, $g$ and $h$, such that $h$ is applied first and $g$ second is given by

$$(g \circ h)(x) = g(h(x))$$

The domain of the composite function $g \circ h$ is the set of all $x$ in the domain of $h$ such that $h(x)$ is in the domain of $g$.

● **Hint:** The notations $(g \circ h)(x)$ and $g(h(x))$ are both commonly used to denote a composite function where $h$ is applied first then followed by applying $g$. Since we are reading this from left to right, it is easy to apply the functions in the incorrect order. It may be helpful to read $g \circ h$ as '$g$ following $h$', or as '$g$ composed with $h$' to emphasize the order in which the functions are applied. Also, in either notation, $(g \circ h)(x)$ or $g(h(x))$, the function applied first is closest to the variable $x$.

## Example 7

If $f(x) = 3x$ and $g(x) = 2x - 6$, find:

a) $(f \circ g)(5)$ b) Express $(f \circ g)(x)$ as a single function rule (expression).

c) $(g \circ f)(5)$ d) Express $(g \circ f)(x)$ as a single function rule (expression).

e) $(g \circ g)(5)$ f) Express $(g \circ g)(x)$ as a single function rule (expression).

### *Solution*

a) $(f \circ g)(5) = f(g(5)) = f(2 \cdot 5 - 6) = f(4) = 3 \cdot 4 = 12$

b) $(f \circ g)(5) = f(g(x)) = f(2x - 6) = 3(2x - 6) = 6x - 18$
Therefore, $(f \circ g)(x) = 6x - 18$
Check with result from a): $(f \circ g)(5) = 6 \cdot 5 - 18 = 30 - 18 = 12$

c) $(g \circ f)(5) = g(f(5)) = g(3 \cdot 5) = g(15) = 2 \cdot 15 - 6 = 24$

d) $(g \circ f)(x) = g(f(x)) = g(3x) = 2(3x) - 6 = 6x - 6$
Therefore, $(g \circ f)(x) = 6x - 6$
Check with result from c): $(g \circ f)(5) = 6 \cdot 5 - 6 = 30 - 6 = 24$

e) $(g \circ g)(5) = g(g(5)) = g(2 \cdot 5 - 6) = g(4) = 2 \cdot 4 - 6 = 2$

f) $(g \circ g)(x) = g(g(x)) = g(2x - 6) = 2(2x - 6) - 6 = 4x - 18$
Therefore, $(g \circ g)(x) = 4x - 18$
Check with result from e): $(g \circ g)(5) = 4 \cdot 5 - 18 = 20 - 18 = 2$

It is important to notice that in parts b) and d) in Example 7, $f \circ g$ is *not* equal to $g \circ f$. At the start of this section, it was shown how the two functions $h(x) = x + 4$ and $g(x) = \sqrt{x}$ could be combined into the composite function $(g \circ h)(x)$ to create the single function $f(x) = \sqrt{x + 4}$. However, the composite function $(h \circ g)(x)$ – the functions applied in reverse order – creates a different function:
$(h \circ g)(x) = h(g(x)) = h(\sqrt{x}) = \sqrt{x} + 4$. Since, $\sqrt{x} + 4 \neq \sqrt{x + 4}$ then again $f \circ g$ is *not* equal to $g \circ f$. Is it always true that $f \circ g \neq g \circ f$? The next example will answer that question.

### Example 8

Given $f: x \mapsto 3x - 6$ and $g: x \mapsto \frac{1}{3}x + 2$, find the following:

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

#### *Solution*

a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x + 2\right) = 3\left(\frac{1}{3}x + 2\right) - 6 = x + 6 - 6 = x$

b) $(g \circ f)(x) = g(f(x)) = g(3x - 6) = \frac{1}{3}(3x - 6) + 2 = x - 2 + 2 = x$

Example 8 shows that it is possible for $f \circ g$ to be equal to $g \circ f$. We will learn in the next section that this occurs in some cases where there is a 'special' relationship between the pair of functions. However, in general $f \circ g \neq g \circ f$.

## Decomposing composite functions

In Examples 7 and 8, we created a single function by forming the composition of two functions. As we did with the function $f(x) = \sqrt{x + 4}$ at the start of this section, it is also important for you to be able to identify two functions that *make up* a composite function, in other words, for you to *decompose* a function into two simpler functions. When you are doing this it is very useful to think of the function which is applied first as the 'inside' function, and the function that is applied second as the 'outside' function. In the function $f(x) = \sqrt{x + 4}$, the 'inside' function is $h(x) = x + 4$ and the 'outside' function is $g(x) = \sqrt{x}$.

• **Hint:** Decomposing composite functions – identifying the component functions that form a composite function – is an important skill when working with certain functions in the topic of calculus. For the composite function $f(x) = (g \circ h)(x)$, $g$ and $h$ are the component functions.

### Example 9

Each of the following functions is a composite function of the form $(f \circ g)(x)$. For each, find the two component functions $f$ and $g$.

a) $h: x \mapsto \dfrac{1}{x + 3}$ b) $k: x \mapsto 2^{4x + 1}$ c) $p(x) = \sqrt[3]{x^2 - 4}$

#### *Solution*

a) If you were to evaluate the function $h(x)$ for a certain $x$ in the domain, you would first evaluate the expression $x + 3$, and then evaluate the expression $\frac{1}{x}$. Hence, the 'inside' function (applied first) is $y = x + 3$, and the 'outside' function (applied second) is $y = \frac{1}{x}$. Then, with $g(x) = x + 3$ and $f(x) = \frac{1}{x}$, it follows that $h: x \mapsto (f \circ g)(x)$.

b) Evaluating $k(x)$ requires you to first evaluate the expression $4x + 1$, and then evaluate the expression $2^x$. Hence, the 'inside' function is $y = 4x + 1$, and the 'outside' function is $y = 2^x$. Then, with $g(x) = 4x + 1$ and $f(x) = 2^x$, it follows that $k: x \mapsto (f \circ g)(x)$.

c) Evaluating $p(x)$ requires you to perform three separate evaluation 'steps': (1) squaring a number, (2) subtracting four, and then (3) taking the cube root. Hence, it is possible to decompose $p(x)$ into three component functions: if $h(x) = x^2$, $g(x) = x - 4$ and $f(x) = \sqrt[3]{x}$, then $p(x) = (f \circ g \circ h)(x) = f(g(h(x)))$. However, for our purposes it is best to decompose the composite function into only two component functions: if $g(x) = x^2 - 4$, and $f(x) = \sqrt[3]{x}$, then $p: x \mapsto (f \circ g)(x) = f(g(x))$.

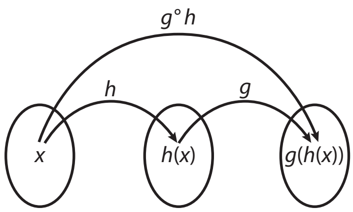



**Figure 2.6**

## Finding the domain of a composition of functions

Referring back to Figure 2.5 (shown again here as Figure 2.6), it is important to note that in order for a value of $x$ to be in the domain of the composite function $g \circ h$, two conditions must be met:

(1) $x$ must be in the domain of $h$, and (2) $h(x)$ must be in the domain of $g$.

Likewise, it is also worth noting that $g(h(x))$ is in the range of $g \circ h$ only if $x$ is in the domain of $g \circ h$. The next example illustrates these points – and also that, in general, the domains of $g \circ h$ and $h \circ g$ are not the same.

### Example 10

Let $g(x) = x^2 - 4$ and $h(x) = \sqrt{x}$. Find:

a) $(g \circ h)(x)$ and its domain and range, and

b) $(h \circ g)(x)$ and its domain and range.

#### *Solution*

Firstly, establish the domain and range for both $g$ and $h$. For $g(x) = x^2 - 4$, the domain is $x \in \mathbb{R}$ and the range is $y \geqslant -4$. For $h(x) = \sqrt{x}$, the domain is $x \geqslant 0$ and the range is $y \geqslant 0$.

a) $(g \circ h)(x) = g(h(x))$

$= g(\sqrt{x})$ To be in the domain of $g \circ h$, $\sqrt{x}$ must be defined for $x \Rightarrow x \geqslant 0$.

$= (\sqrt{x})^2 - 4$ Therefore, the domain of $g \circ h$ is $x \geqslant 0$.

$= x - 4$ Since $x \geqslant 0$, the range for $y = x - 4$ is $y \geqslant -4$.

Therefore, $(g \circ h)(x) = x - 4$, and its domain is $x \geqslant 0$, and its range is $y \geqslant -4$.

b) $(h \circ g)(x) = h(g(x))$ $g(x) = x^2 - 4$ must be in the domain of $h$ $h \Rightarrow x^2 - 4 \geqslant 0 \Rightarrow x^2 \geqslant 4$.

$= h(x^2 - 4)$ Therefore, the domain of $h \circ g$ is $x \leqslant -2$ or $x \geqslant 2$

$= \sqrt{x^2 - 4}$ and with $x \leqslant -2$ or $x \geqslant 2$, the range for $y = \sqrt{x^2 - 4}$ is $y \geqslant 0$.

Therefore, $(h \circ g)(x) = \sqrt{x^2 - 4}$, and its domain is $x \leqslant -2$ or $x \geqslant 2$, and its range is $y \geqslant 0$.

**Exercise 2.2**

**1** Let $f(x) = 2x$ and $g(x) = \frac{1}{x-3}$, $x \neq 0$.

a) Find the value of (i) $(f \circ g)(5)$ and (ii) $(g \circ f)(5)$.
b) Find the function rule (expression) for (i) $(f \circ g)(x)$ and (ii) $(g \circ f)(x)$.

**2** Let $f: x \mapsto 2x - 3$ and $g: x \mapsto 2 - x^2$.
In a-f, evaluate:
a) $(f \circ g)(0)$ b) $(g \circ f)(0)$ c) $(f \circ f)(4)$
d) $(g \circ g)(-3)$ e) $(f \circ g)(-1)$ f) $(g \circ f)(-3)$

In g-j, find the expression:
g) $(f \circ g)(x)$ h) $(g \circ f)(x)$ i) $(f \circ f)(x)$ j) $(g \circ g)(x)$

For each pair of functions in 3–7, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and state the domain for each.

**3** $f(x) = 4x - 1$, $g(x) = 2 + 3x$

**4** $f(x) = x^2 + 1$, $g(x) = -2x$

**5** $f(x) = \sqrt{x+1}$, $g(x) = 1 + x^2$

**6** $f(x) = \frac{2}{x+4}$, $g(x) = x - 1$

**7** $f(x) = 3x + 5$, $g(x) = \frac{x-5}{3}$

**8** Let $g(x) = \sqrt{x-1}$ and $h(x) = 10 - x^2$. Find:
a) $(g \circ h)(x)$ and its domain and range
b) $(h \circ g)(x)$ and its domain and range.

In 9–14, determine functions $g$ and $h$ so that $f(x) = g(h(x))$.

**9** $f(x) = (x+3)^2$

**10** $f(x) = \sqrt{x-5}$

**11** $f(x) = 7 - \sqrt{x}$

**12** $f(x) = \frac{1}{x+3}$

**13** $f(x) = 10^{x+1}$

**14** $f(x) = \sqrt[3]{x-9}$

In 15-18, find the domain for a) the function $f$, b) the function $g$, and c) the composite function $f \circ g$.

**15** $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

**16** $f(x) = \frac{1}{x}$, $g(x) = x + 3$

**17** $f(x) = \frac{3}{x^2 - 1}$, $g(x) = x + 1$

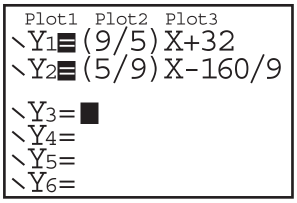
**18** $f(x) = 2x + 3$, $g(x) = \frac{x}{2}$

# 2.3 Inverse functions

## Pairs of inverse functions

Let's look again at the function at the start of this chapter – the formula that converts degrees Celsius ($C$) to degrees Fahrenheit ($F$): $F = \frac{9}{5}C + 32$. If we rearrange the function so that $C$ is the independent variable (i.e. $C$ is expressed in terms of $F$), we get a different formula that does the reverse, or *inverse* process, andconverts $F$ to $C$. Writing $C$ in terms of $F$ (solving for $C$) gives: $C = \frac{5}{9}(F - 32)$or $C = \frac{5}{9}F - \frac{160}{9}$. This new formula could be useful for people travelling to the USA. These two conversion formulas, $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}F - \frac{160}{9}$, are both

**Hint:** Writing a function using $x$ and $y$ for the independent and dependent variables, such that $y$ is expressed in terms of $x$, is a good idea because this is the format in which you must enter it on your GDC in order to have the GDC display a graph or table for the function.




linear functions. As mentioned previously, it is typical for the independent variable (domain) of a function to be $x$ and the dependent variable (range) to be $y$. Let's assign the name $f$ to the function converting $C$ to $F$, and the name $g$ to the function converting $F$ to $C$.

$$\text{converting } C \text{ to } F: \quad y = \frac{9}{5}x + 32 \quad \Rightarrow \quad f(x) = \frac{9}{5}x + 32$$

$$\text{converting } C \text{ to } F: \quad y = \frac{5}{9}x - \frac{160}{9} \quad \Rightarrow \quad g(x) = \frac{5}{9}x - \frac{160}{9}$$

The two functions, $f$ and $g$, have a 'special' relationship in that they 'undo' each other.

To illustrate, function $f$ converts $25\,°C$ to $77\,°F$ $\left[f(25) = \frac{9}{5}(25) + 32 = 45 + 32 = 77\right]$, and then function $g$ can 'undo' this by converting $77\,°F$ back to $25\,°C$ $\left[g(77) = \frac{5}{9}(77) - \frac{160}{9} = \frac{385 - 160}{9} = \frac{225}{9} = 25\right]$. Because function $g$ has this reverse (inverse) effect on function $f$, we call function $g$ the *inverse* of function $f$. Function $f$ has the same inverse effect on function $g$ [$g(77) = 25$ and then $f(25) = 77$], making $f$ the inverse function of $g$. The functions $f$ and $g$ are inverses of each other – they are a *pair of inverse functions*.

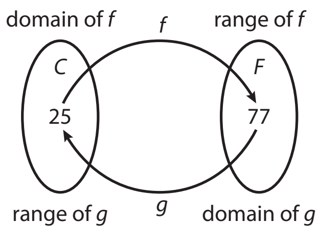



**Figure 2.7**

In Figure 2.7, the mapping diagram for the functions $f$ and $g$ illustrates the inverse relationship for a pair of inverse functions where the domain of one is the range for the other.

## The composition of two inverse functions

The mapping diagram (Figure 2.7) and the numerical examples in the previous paragraph indicate that if function $f$ is applied to a number in its domain (e.g. 25) giving a result in the range of $f$ (i.e. 77) and then function $g$ is applied to this result, the final result (i.e. 25) is the same number first chosen from the domain of $f$. This process and result can be expressed symbolically as: $(g \circ f)(x) = x$ or $g(f(x)) = x$. The composition of two inverse functions maps any value $x$ back to itself – i.e. one function 'undoing' the other. It must also follow that $(f \circ g) = x$. Let's verify these results for the pair of inverse functions $f$ and $g$.

$$(g \circ f)(x) = g\left(\frac{9}{5}x + 32\right) = \frac{5}{9}\left(\frac{9}{5}x + 32\right) - \frac{160}{9} = x + \frac{160}{9} - \frac{160}{9} = x$$

$$f(g(x)) = f\left(\frac{5}{9}x - \frac{160}{9}\right) = \frac{9}{5}\left(\frac{5}{9}x - \frac{160}{9}\right) + 32 = x - \frac{160}{5} + 32$$

$$= x - 32 + 32 = x$$

You are already familiar with pairs of **inverse operations**. Addition and subtraction are inverse operations. For example, the rule of 'adding six' ($x + 6$), and the rule of 'subtracting six' ($x - 6$) undo each other. Accordingly, the functions $f(x) = x + 6$ and $g(x) = x - 6$ are a pair of inverse functions. Multiplication and division are also inverse operations.

Examples 7 and 8 in the previous section on composite functions explored whether $f \circ g = g \circ f$. Example 7 provided a counter example showing it is not a true statement. However, Example 8 showed a pair of functions for which $(f \circ g)(x) = (g \circ f)(x) = x$; the same result that we just obtained for the pair of inverse functions that convert between $C$ and $F$. The two functions in Example 8, $f: x \mapsto 3x - 6$ and $g: x \mapsto \frac{1}{3}x + 2$, are also a pair of inverse functions.

**Definition of the inverse of a function**

If $f$ and $g$ are two functions such that $(f \circ g)(x) = x$ for every $x$ in the domain of $g$ and $(g \circ f)(x) = x$ for every $x$ in the domain of $f$, the function $g$ is the *inverse* of the function $f$. The notation to indicate the function that is the 'inverse of function $f$' is $f^{-1}$. Therefore,

$$(f \circ f^{-1})(x) = x \text{ and } (f^{-1} \circ f)(x) = x$$

The domain of $f$ must be equal to the range of $f^{-1}$, and the range of $f$ must be equal to the domain of $f^{-1}$.

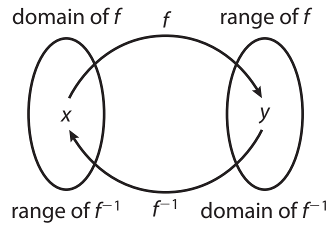



**Figure 2.8** $f(x) = y$ and $f^{-1}(y) = x$

Figure 2.7 shows a mapping diagram for a pair of inverse functions.

It follows from the definition that if $g$ is the inverse of $f$, it must also be true that $f$ is the inverse of $g$.

• **Hint:** Do not mistake the $-1$ in the notation $f^{-1}$ for an exponent. It is *not* an exponent. $f^{-1}$ does *not* denote the reciprocal of $f(x)$. If a superscript of $-1$ is applied to the name of a function – as in $f^{-1}(x)$ or $\sin^{-1}(x)$ – it denotes the function that is the inverse of the named function (e.g. $f(x)$ or $\sin(x)$). If a superscript of $-1$ is applied to an expression, as in $7^{-1}$ or $(2x + 5)^{-1}$ or $(f(x))^{-1}$, it is an exponent and denotes the reciprocal of the expression. For example, the reciprocal of $f(x)$ is $(f(x))^{-1} = \frac{1}{f(x)}$.

## Finding the inverse of a function

### Example 11

Given the linear function $f(x) = 4x - 8$, find its inverse function $f^{-1}(x)$ and verify the result by showing that $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x) = x$.

#### *Solution*

Recall that the way we found the inverse of the function converting $C$ to $F$, $F = \frac{9}{5}C + 32$, was by making the independent variable the dependent variable and vice versa. Essentially what we are doing is switching the domain ($x$) and range ($y$) since the domain of $f$ becomes the range of $f^{-1}$ and the range of $f$ becomes the domain of $f^{-1}$, as stated in the definition of the inverse of the function, and depicted in Figure 2.8. Also, recall that $y = f(x)$.

$f(x) = 4x - 8$

$y = 4x - 8$ write $y = f(x)$

$x = 4y - 8$ interchange $x$ and $y$ (i.e. switch the domain and range)

$4y = x + 8$ solve for $y$ (dependent variable) in terms of $x$ (in dependent variable)

$y = \frac{1}{4}x + 2$

$f^{-1}(x) = \frac{1}{4}x + 2$ resulting equation is $y = f^{-1}(x)$

Verify that $f$ and $f^{-1}$ are inverses by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$$f\left(\tfrac{1}{4}x + 2\right) = 4\left(\tfrac{1}{4}x + 2\right) - 8 = x + 8 - 8 = x$$

$$f^{-1}(4x - 8) = \tfrac{1}{4}(4x - 8) + 2 = x - 2 + 2 = x$$

This confirms that $y = 4x - 8$ and $y = \frac{1}{4}x + 2$ are inverses of each other.

The method of interchanging $x$ and $y$ to find the inverse function also gives us a way for obtaining the graph of $f^{-1}$ from the graph of $f$. Given the reversing effect that a pair of inverse functions have on each other, if $f(a) = b$ then $f^{-1}(b) = a$. Hence, if the ordered pair $(a, b)$ is a point on the graph of $y = f(x)$, the 'reversed' ordered pair $(b, a)$ must be on the graph of $y = f^{-1}(x)$. Figure 2.9 shows that the point $(b, a)$ can be found by reflecting the point $(a, b)$ about the line $y = x$.

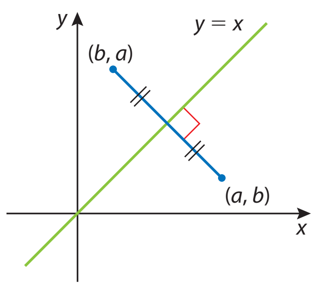



**Figure 2.9**

As Figure 2.10 illustrates, the following is true.

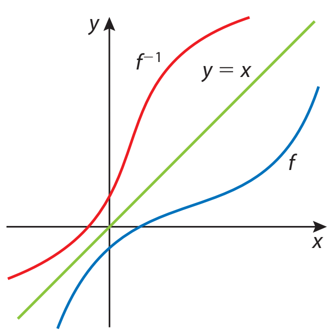



**Figure 2.10**

**Graphical symmetry of inverse functions**

The graph of $f^{-1}$ is a reflection of the graph of $f$ about the line $y = x$.

## The identity function

We have repeatedly demonstrated the fact, and it is formally stated in the definition of the inverse of a function, that the composite function which has a pair of inverse functions as its components is always the linear function $y = x$. That is, $(f \circ f^{-1})(x) = x$ or $(f^{-1} \circ f)(x) = x$. Let's label the function $y = x$ with the name $I$. Along with the fact that $I(x) = (f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, the function $I(x)$ has two other interesting properties. It is obvious that the line $y = x$ is reflected back to itself when reflected about the line $y = x$.

Hence, from the graphical symmetry of inverse functions, the function $I(x)$ is its own inverse; that is, $I(x) = I^{-1}(x)$. Most interestingly, $I(x)$ behaves in composite functions just like the number one behaves for real numbers and multiplication. The number one is the **identity element** for multiplication. For any function $f$, it is true that $f \circ I = f$ and $I \circ f = f$. For this reason, we call the function $f(x) = x$, or $I(x) = x$, the **identity function**.

When $f(x) = f^{-1}(x)$ the function $f$ is said to be **self-inverse**. The fact that the function $f(x) = x$ is self-inverse should make you wonder if there are any other functions with the same property. Knowing that inverses are symmetric about the line $y = x$, we only need to find a function whose graph has $y = x$ as a line of symmetry.

## The existence of an inverse function

Is it possible for the inverse of a function not to be a function? Recall that the definition of a function (Section 2.1) says that a function is a relation such that a certain value $x$ in the domain produces only one value $y$ in the range. The vertical line test for functions followed from this definition.

### Example 12

Find the inverse of the function $g(x) = x^2 + 2$ with domain $x \in \mathbb{R}$.

***Solution***

Following the method used in Example 11:

$$g(x) = x^2 + 2$$
$$y = x^2 + 2$$
$$x = y^2 + 2$$
$$y^2 = x - 2$$
$$y = \pm\sqrt{x-2}$$

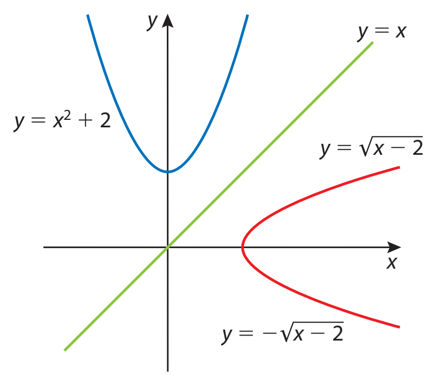



Figure 2.11

Certainly the graphs of $y = x^2 + 2$ and $y = \pm\sqrt{x-2}$ are reflections about the line $y = x$ (see Figure 2.11). However, the graph of $y = \pm\sqrt{x-2}$ does not pass the vertical line test. $y = \pm\sqrt{x-2}$ is the inverse of $g(x) = x^2 + 2$, but it is only a relation and *not* a function.

The inverse of $g(x)$ will be a function only if $g(x)$ is a one-to-one function; that is, a function such that no two elements in the domain $(x)$ of $g$ correspond to the same element in the range $(y)$. The graph of a one-to-one function must pass both a vertical line test and also a horizontal line test.

The function $f(x) = x^2$ with domain $x \in \mathbb{R}$ (Figure 2.12) is *not* a one-to-one function. Hence, its inverse is *not* a function. There are two different values of $x$ that correspond to the same value of $y$; for example, $x = 2$ and $x = -2$ both get mapped to $y = 4$. Hence, $f$ does *not* pass the horizontal line test.

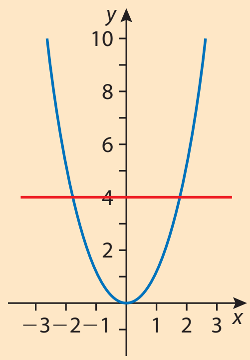



**Figure 2.12**

The function $f(x) = x^2$ with domain $x \geqslant 0$ is a one-to-one function (Figure 2.13). Hence, its inverse is also a function. [Note: domain changed to $x \geqslant 0$.]

A function $f$ has an inverse function $f^{-1}$ if and only if $f$ is one-to-one.

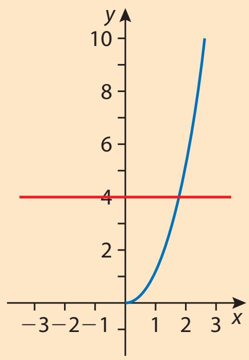



**Figure 2.13**

**Definition of a one-to-one function**

A function is one-to-one if each element $y$ in the range is the image of exactly one element $x$ in the domain. No horizontal line can pass through the graph of a one-to-one function at more than one point (horizontal line test).

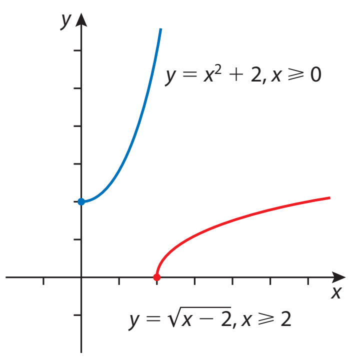
Referring back to Example 12, you now understand that the function $g(x) = x^2 + 2$ with domain $x \in \mathbb{R}$ does not have an inverse function $g^{-1}(x)$. However, if the domain is changed so that $g(x)$ is one-to-one, then $g^{-1}(x)$ exists. There is not only one way to change the domain of a function in order to make it one-to-one.

## Example 13

Given $g(x) = x^2 + 2$ such that $x \geqslant 0$, find $g^{-1}(x)$ and state its domain.

### *Solution*

Given that the domain is $x \geqslant 0$, then the range for $g(x)$ will be $y \geqslant 2$. Since the domain and range are switched for the inverse, for $g^{-1}(x)$ the domain is $x \geqslant 2$ and the range is $y \geqslant 2$. Given the working in Example 12, it follows that $g^{-1}(x) = \sqrt{x - 2}$ with domain $x \geqslant 2$.

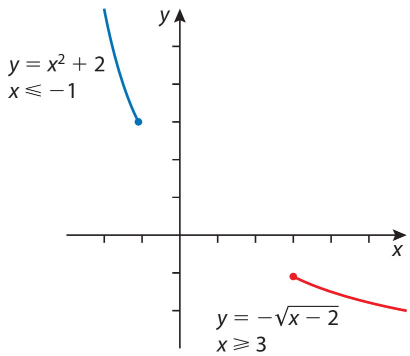


## Example 14

Given $g(x) = x^2 + 2$ such that $x \leqslant -1$, find $g^{-1}(x)$ and state its domain.

### Solution

Given that the domain is $x \leqslant -1$, then the range for $g(x)$ will be $y \geqslant 3$. Since the domain and range are switched for the inverse, for $g^{-1}(x)$ the domain is $x \geqslant 3$ and the range is $y \leqslant -1$. Given the working in Example 12, it follows that $g^{-1}(x) = -\sqrt{x-2}$ with domain $x \geqslant 3$.




● **Examiner's Hint:** For the Mathematics Standard Level course, if an inverse function is to be found, the given function will be defined with a domain that ensures it is one-to-one.

**Finding the inverse of a function**

To find the inverse of a function *f*, use the following steps:

**1** Confirm that *f* is one-to-one (although, for this course, you can assume this).
**2** Replace $f(x)$ with $y$.
**3** Interchange $x$ and $y$.
**4** Solve for $y$.
**5** Replace $y$ with $f^{-1}(x)$.
**6** The domain of $f^{-1}$ is equal to the range of *f*, and the range of $f^{-1}$ is equal to the domain of *f*.

## Example 15

Consider the function $f: x \mapsto \sqrt{x+3}, x \geqslant -3$.

a) Determine the inverse function $f^{-1}$.

b) What is the domain of $f^{-1}$?

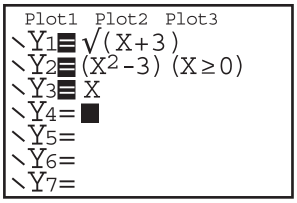
### Solution

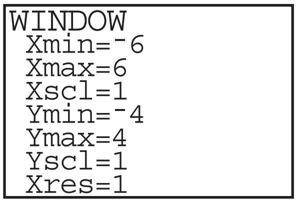
a) Following the steps for finding the inverse of a function gives:

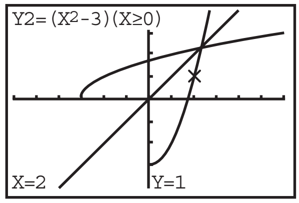
$$y = \sqrt{x+3} \qquad \text{replace } f(x) \text{ with } y$$
$$x = \sqrt{y+3} \qquad \text{interchange } x \text{ and } y$$
$$x^2 = y + 3 \qquad \text{solve for } y\text{; squaring both sides here}$$
$$y = x^2 - 3 \qquad \text{solved for } y$$
$$f^{-1}: x \mapsto x^2 - 3 \qquad \text{replace } y \text{ with } f^{-1}(x)$$

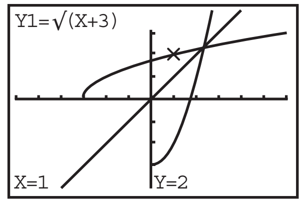
b) The domain explicitly defined for *f* is $x \geqslant -3$ and since the $\sqrt{\ }$ symbol stands for the principal square root (positive), then the range of *f* is all positive real numbers, i.e. $y \geqslant 0$. The domain of $f^{-1}$ is equal to the range of *f*, therefore, the domain of $f^{-1}$ is $x \geqslant 0$.

Graphing $y = \sqrt{x+3}$ and $y = x^2 - 3$ from Example 15 on your GDC visually confirms these results. Note that since the calculator would have automatically assumed that the domain is $x \in \mathbb{R}$, the domain for the equation $y = x^2 - 3$ has been changed to $x \geqslant -3$. In order to show that $f$ and $f^{-1}$ are reflections about the line $y = x$, the line $y = x$ has been graphed and a viewing window has been selected to ensure that the scales are equal on each axis. Using the trace feature of your GDC, you can explore a characteristic of inverse functions – that is, if some point $(a, b)$ is on the graph of $f$, the point $(b, a)$ must be on the graph of $f^{-1}$.










## Example 16

Consider the function $f(x) = 2(x + 4)$ and $g(x) = \frac{1 - x}{3}$.

a) Find $g^{-1}$ and state its domain and range.

b) Solve the equation $(f \circ g^{-1})(x) = 2$

### *Solution*

a)

| | |
|---|---|
| $y = \frac{1 - x}{3}$ | replace $f(x)$ with $y$ |
| $x = \frac{1 - y}{3}$ | interchange $x$ and $y$ |
| $3x = 1 - y$ | solve for $y$ |
| $y = -3x + 1$ | solved for $y$ |
| $g^{-1}(x) = -3x + 1$ | replace $y$ with $g^{-1}(x)$ |

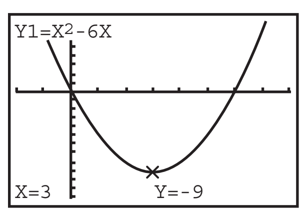
$g$ is a linear function and its domain is $x \in \mathbb{R}$ and its range is $y \in \mathbb{R}$; therefore, for $g^{-1}$ the domain is $x \in \mathbb{R}$ and range is $y \in \mathbb{R}$.

b)

$$(f \circ g^{-1})(x) = f(g^{-1}(x)) = f(-3x + 1) = 2$$
$$2[(-3x + 1) + 4] = 2$$
$$-6x + 2 + 8 = 2$$
$$-6x = -8$$
$$x = \tfrac{4}{3}$$

## Example 17

Given $f(x) = x^2 - 6x$, find the inverse $f^{-1}(x)$ and state its domain.




### *SolutioAn*

The graph of $f(x) = x^2 - 6x$, $x \in \mathbb{R}$, is a parabola with a vertex at $(3, -9)$. It is not a one-to-one function. There are many ways to restrict the domain of $f$ to make it one-to-one. The choices that have the domain as large as possible are $x \geqslant 3$ or $x \leqslant 3$. Let's change the domain of $f$ to $x \geqslant 3$.

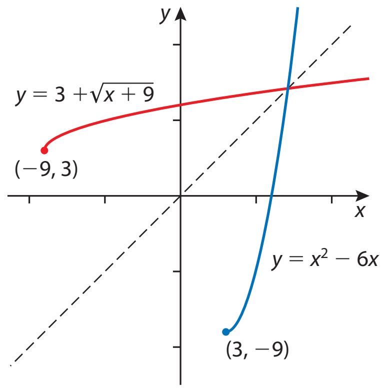


**Figure 2.14**

$y = x^2 - 6x$ replace $f(x)$ with $y$

$x = y^2 - 6y$ interchange $x$ and $y$

$y^2 - 6y + 9 = x + 9$ solve for $y$ by 'completing the square'

$(y - 3)^2 = x + 9$

$y - 3 = \pm\sqrt{x + 9}$

$y = 3 + \sqrt{x + 9}$ + rather than ± because range of $f^{-1}$ is $x \geqslant 3$ (domain of $f$)

In order for $\sqrt{x+9}$ to be a real number then $x \geqslant -9$.

Therefore, $f^{-1}(x) = 3 + \sqrt{x+9}$ and the domain of $f^{-1}$ is $x \geqslant -9$.

The inverse relationship between $f(x) = x^2 - 6x$ and $f^{-1}(x) = 3 + \sqrt{x+9}$ is confirmed graphically in Figure 2.14.

## Exercise 2.3

In questions 1–4, assume that $f$ is a one-to-one function.

**1** a) If $f(2) = -5$, what is $f^{-1}(-5)$? b) If $f^{-1}(6) = 10$, what is $f(10)$?

**2** a) If $f(-1) = 13$, what is $f^{-1}(13)$? b) If $f^{-1}(b) = a$, what is $f(a)$?

**3** If $g(x) = 3x - 7$, what is $g^{-1}(5)$?

**4** If $h(x) = x^2 - 8x$, with $x \geqslant 4$, what is $f^{-1}(-12)$?

In questions 5–12, show a) algebraically and b) graphically that $f$ and $g$ are inverse functions by verifying that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, and by sketching the graphs of $f$ and $g$ on the same set of axes with equal scales on the $x$- and $y$-axes. Use your GDC to assist in making your sketches on paper.

**5** $f: x \mapsto x + 6; g: x \mapsto x - 6$

**6** $f: x \mapsto 4x; g: x \mapsto \frac{x}{4}$

**7** $f: x \mapsto 3x + 9; g: x \mapsto \frac{1}{3}x - 3$

**8** $f: x \mapsto \frac{1}{x}, g: x \mapsto \frac{1}{x}$

**9** $f: x \mapsto x^2 - 2, x \geqslant 0; g: x \mapsto \sqrt{x+2}, x \geqslant -2$

**10** $f: x \mapsto x^3, g: x \mapsto \sqrt[3]{x}$

**11** $f: x \mapsto \frac{1}{1+x}, g: x \mapsto \frac{1-x}{x}$

**12** $f: x \mapsto (6 - x)^{\frac{1}{2}}; g: x \mapsto 6 - x^2, x \geqslant 0$

In questions 13–20, find the inverse function $f^{-1}$ and state its domain.

**13** $f(x) = 2x - 3$

**14** $f(x) = \frac{x+7}{4}$

**15** $f(x) = \sqrt{x}$

**16** $f(x) = \frac{1}{x+2}$

**17** $f(x) = 4 - x^2, x \geqslant 0$

**18** $f(x) = \sqrt{x-5}$

**19** $f(x) = ax + b, a \neq 0$

**20** $f(x) = x^2 + 2x, x \geqslant -1$

In questions 21–28, use the functions $g(x) = x + 3$ and $h(x) = 2x - 4$ to find the indicated value or the indicated function.

**21** $(g^{-1} \circ h^{-1})(5)$

**22** $(h^{-1} \circ g^{-1})(9)$

**23** $(g^{-1} \circ g^{-1})(2)$

**24** $(h^{-1} \circ h^{-1})(2)$

**25** $g^{-1} \circ h^{-1}$

**26** $h^{-1} \circ g^{-1}$

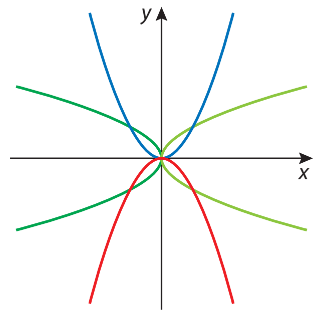
**27** $(g \circ h)^{-1}$

**28** $(h \circ g)^{-1}$

**29** The function in question 8, $f(x) = \frac{1}{x}$, is its own inverse (self-inverse). Show that any function in the form $f(x) = \frac{a}{x+b} - b, a \neq 0$ is its own inverse.

# 2.4 Transformations of functions

Even when you use your GDC to sketch the graph of a function, it is helpful to know what to expect in terms of the location and shape of the graph – and even more so if you're not allowed to use your GDC for a particular question. In this section, we look at how certain changes to the equation of a function can affect, or **transform**, the location and shape of its graph. We will investigate three different types of **transformations** of functions that include how the graph of a function can be **translated, reflected** and **stretched** (or shrunk). This will give us a better understanding of how to efficiently sketch and visualize many different functions.



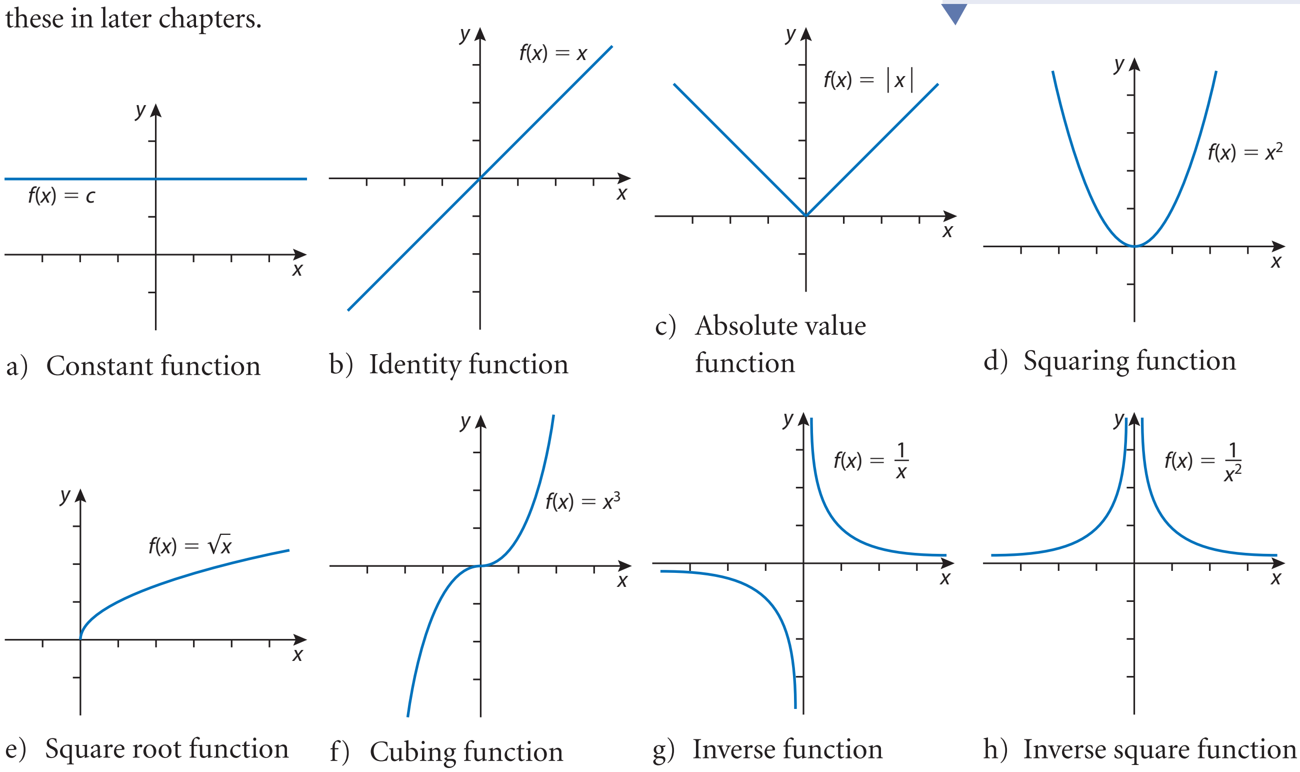

• **Hint:** When analyzing the graph of a function, it is often convenient to express a function in the form $y = f(x)$. As we have done throughout this chapter, we often refer to a function such as $f(x) = x^2$ by the equation $y = x^2$.

## Graphs of common functions

It is important for you to be familiar with the location and shape of a certain set of common functions. For example, from your previous knowledge about linear equations, you can determine the location of the linear function $f(x) = ax + b$. You know that the graph of this function is a line whose slope is $a$ and whose $y$-intercept is $(0, b)$.

The eight graphs in Figure 2.15 represent some of the most commonly used functions in algebra. You should be familiar with the characteristics of the graphs of these common functions. This will help you predict and analyze the graphs of more complicated functions that are derived from applying one or more transformations to these simple functions. There are other important basic functions with which you should be familiar – for example, exponential, logarithmic and exponential functions – but we will encounter these in later chapters.

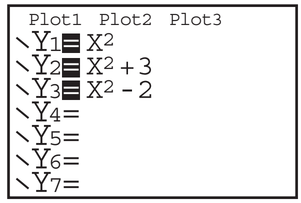
**Figure 2.15** Common functions

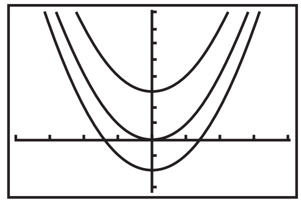


● **Hint:** The word *inverse* can have different meanings in mathematics depending on the context. In Section 2.3 of this chapter, *inverse* is used to describe operations or functions that undo each other. However, 'inverse' is sometimes used to denote the **multiplicative inverse** (or **reciprocal**) of a number or function. This is how it is used in the names for the functions shown in (g) and (h) of Figure 2.15. The function in g) is sometimes called the **reciprocal function**.

We will see that many functions have graphs that are a transformation (translation, reflection or stretch), or a combination of transformations, of one of these common functions.

## Vertical and horizontal translations



Use your GDC to graph each of the following three functions: $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 2$. How do the graphs of $g$ and $h$ compare with the graph of $f$ that is one of the common functions displayed in Figure 2.15? The graphs of $g$ and $h$ both appear to have the same shape – it's only the location, or position, that has changed compared to $f$. Although the curves (parabolas) appear to be getting closer together, their vertical separation at every value of $x$ is constant.

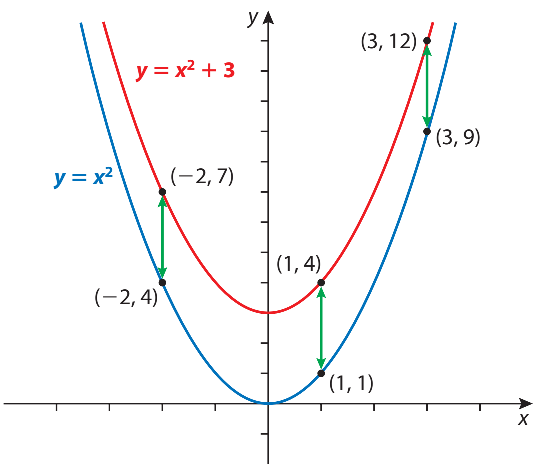



Figure 2.16

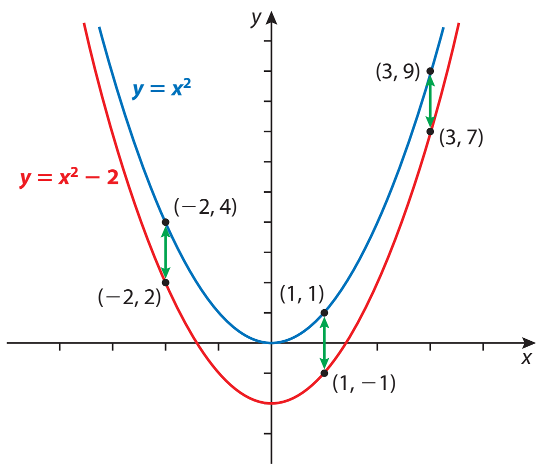



Figure 2.17

As Figures 2.16 and 2.17 clearly show, you can obtain the graph of $g(x) = x^2 + 3$ by translating (shifting) the graph of $f(x) = x^2$ *up* three units, and you can obtain the graph of $h(x) = x^2 - 2$ by translating the graph of $f(x) = x^2$ *down* two units.

**Vertical translations of a function**

Given $k > 0$, then:

I. The graph of $y = f(x) + k$ is obtained by translating *up* $k$ units the graph of $y = f(x)$.

II. The graph of $y = f(x) - k$ is obtained by translating *down* $k$ units the graph of $y = f(x)$.

Change function $g$ to $g(x) = (x + 3)^2$ and change function $h$ to $h(x) = (x - 2)^2$. Graph these two functions along with the 'parent' function

$f(x) = x^2$ on your GDC. This time we observe that the functions $g$ and $h$ can be obtained by a horizontal translation of $f$.

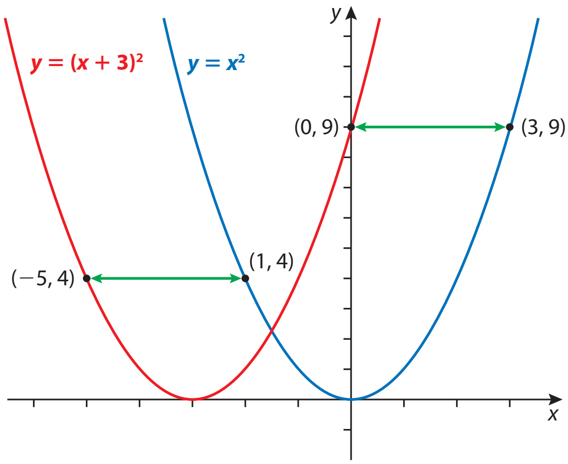
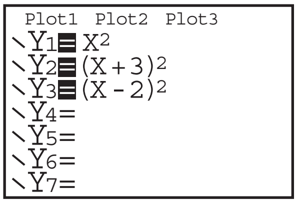
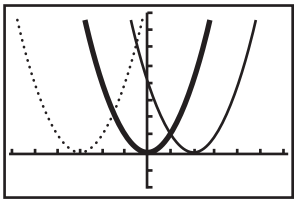



Figure 2.18




Note that a different graphing style is assigned to each equation on the GDC.



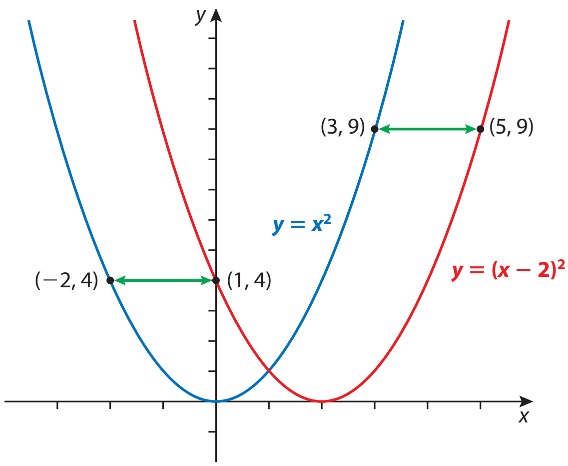



Figure 2.19

As Figures 2.18 and 2.19 clearly show, you can obtain the graph of $g(x) = (x + 3)^2$ by translating the graph of $f(x) = x^2$ three units to the *left*, and you can obtain the graph of $h(x) = (x - 2)^2$ by translating the graph of $f(x) = x^2$ two units to the *right*.

**Horizontal translations of a function**

Given $k > 0$, then:

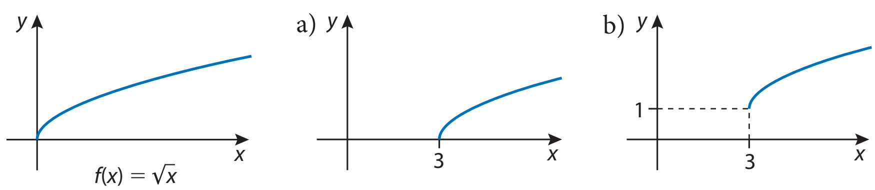
I. The graph of $y = f(x - h)$ is obtained by translating the graph of $y = f(x)$ $h$ units to the *right*.

II. The graph of $y = f(x + h)$ is obtained by translating the graph of $y = f(x)$ $h$ units to the *left*.

● **Hint:** A common error is caused by confusion about the direction of a horizontal translation since $f(x)$ is translated *left* if a *positive* number is added *inside* the argument of the function – e.g. $g(x) = (x + 3)^2$ is obtained by translating $f(x) = x$ three units *left*. You are in the habit of associating *positive* with movement to the *right* (as on the $x$-axis) instead of *left*. Whereas $f(x)$ is translated *up* if a *positive* number is added *outs ide* the function – e.g. $g(x) = x^2 + 3$ is obtained by translating $f(x) = x$ three units *up*. This agrees with the convention that a *positive* number is associated with an *upward* movement (as on the $y$-axis). An alternative (and more consistent) approach to vertical and horizontal translations is to think of what number is being added directly to the $x$- or $y$-coordinate. For example, the equation for the graph obtained by translating the graph of $y = x^2$ three units up is $y = x^2 + 3$, which can also be written as $y - 3 = x^2$. In this form, negative three is added to the $y$-coordinate (vertical coordinate), which causes a vertical translation in the *upward* (or positive) direction. Likewise, the equation for the graph obtained by translating the graph of $y = x^2$ two units to the right is $y = (x - 2)^2$. Negative two is added to the $x$-coordinate (horizontal coordinate), which causes a horizontal translation to the right (or positive direction). There is consistency between vertical and horizontal translations. Assuming that movement up or to the right is considered positive, and that movement down or to the left is negative, then the direction for either type of translation is opposite to the sign (±) of the number being added to the vertical ($y$) or horizontal ($x$) coordinate. In fact, what is actually being translated is the $y$-axis or the $x$-axis. For example, the graph of $y - 3 = x^2$ can also be obtained by not changing the graph of $y = x^2$ but instead translating the $y$-axis three units down – which creates exactly the same effect as translating the graph of $y = x^2$ three units up.

## Example 18

The diagrams show how the graph of $y = \sqrt{x}$ is transformed to the graph of $y = f(x)$ in three steps. For each diagram, a) and b), give the equation of the curve.



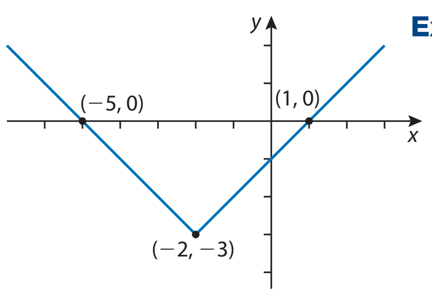

Note that in Example 18, if the transformations had been performed in reverse order – that is, the vertical translation followed by the horizontal translation – it would produce the same final graph (in part (b)) with the same equation. In other words, when applying both a vertical and horizontal translation on a function it does not make any difference which order they are applied (i.e. they are commutative). However, as we will see further on in the chapter, it *can* make a difference how other sequences of transformations are applied. In general, transformations are *not* commutative.

### *Solution*

To obtain graph a), the graph of $y = \sqrt{x}$ is translated three units to the right. To produce the equation of the translated graph, $-3$ is added *inside* the argument of the function $y = \sqrt{x}$. Therefore, the equation of the curve graphed in a) is $y = \sqrt{x - 3}$.

To obtain graph b), the graph of $y = \sqrt{x - 3}$ is translated up one unit. To produce the equation of the translated graph, $+1$ is added *outside* the function. Therefore, the equation of the curve graphed in b) is $y = \sqrt{x - 3} + 1$ (or $y = 1 + \sqrt{x - 3}$).

## Example 19



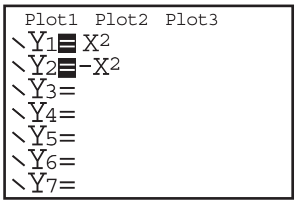

Write the equation of the absolute value function whose graph is shown on the left.

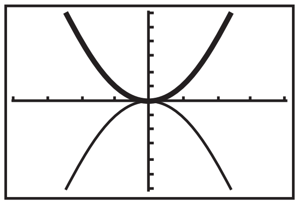
### *Solution*

The graph shown is exactly the same shape as the graph of the equation $y = |x|$ but in a different position. Given that the vertex is $(-2, -3)$, it is clear that this graph can be obtained by translating $y = |x|$ two units left and then three units down. When we move $y = |x|$ two units left we get the graph of $y = |x + 2|$. Moving the graph of $y = |x + 2|$ three units down gives us the graph of $y = |x + 2| - 3$. Therefore, the equation of the graph shown is $y = |x + 2| - 3$. (Note: The two translations applied in reverse order produce the same result.)

## Reflections

Use your GDC to graph the two functions $f(x) = x^2$ and $g(x) = -x^2$. The graph of $g(x) = -x^2$ is a reflection in the $x$-axis of $f(x) = x^2$. This certainly makes sense because $g$ is formed by multiplying $f$ by $-1$ causing the $y$-coordinate of each point on the graph of $y = -x^2$ to be the negative of the $y$-coordinate of the point on the graph of $y = x^2$ with the same $x$-coordinate.



Figures 2.20 and 2.21 illustrate that the graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the $x$-axis.

• **Hint:** The expression $-x^2$ is potentially ambiguous. It is accepted to be equivalent to $-(x)^2$. It is *not* equivalent to $(-x)^2$. For example, if you enter the expression $-3^2$ into your GDC, it gives a result of $-9$, *not* $+9$. In other words, the expression $-3^2$ is consistently interpreted as $3^2$ being multiplied by $-1$. The same as $-x^2$ is interpreted as $x^2$ being multiplied by $-1$.

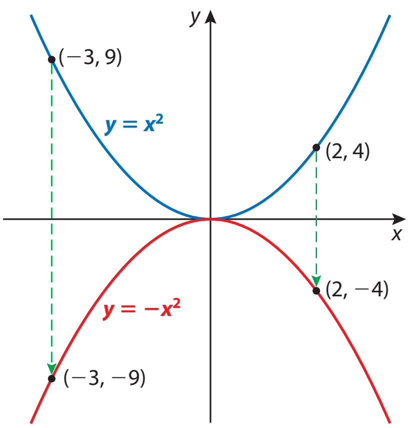



**Figure 2.20**

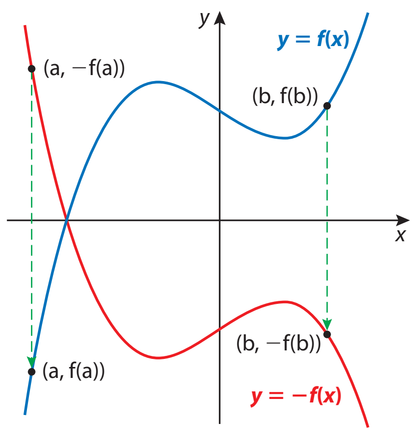



**Figure 2.21**

Graph the functions $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$. Previously, with $f(x) = x^2$ and $g(x) = -x^2$, $g$ was formed by multiplying the entire function $f$ by $-1$. However, for $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$, $g$ is formed by multiplying the variable $x$ by $-1$. In this case, the graph of $g(x) = \sqrt{-x - 2}$ is a reflection in the $y$-axis of $f(x) = \sqrt{x - 2}$. This makes sense if you recognize that the $y$-coordinate on the graph of $y = \sqrt{-x}$ will be the same as the $y$-coordinate on the graph of $y = \sqrt{x}$ if the value substituted for $x$ in $y = \sqrt{-x}$ is the opposite of the $x$ value in $y = \sqrt{x}$. For example, if $x = 9$ then $y = \sqrt{9} = 3$; and, if $x = -9$ then $y = \sqrt{-(-9)} = \sqrt{9} = 3$. Opposite values of $x$ in the two functions produce the same $y$-coordinate for each.

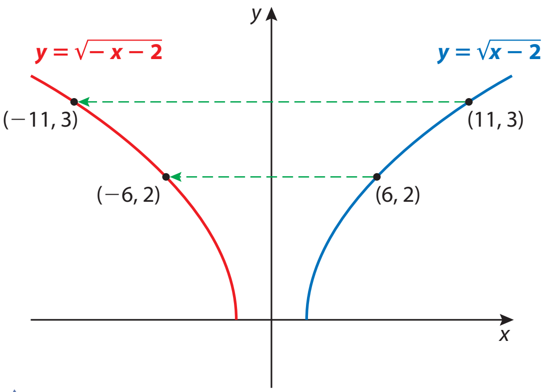



**Figure 2.22**

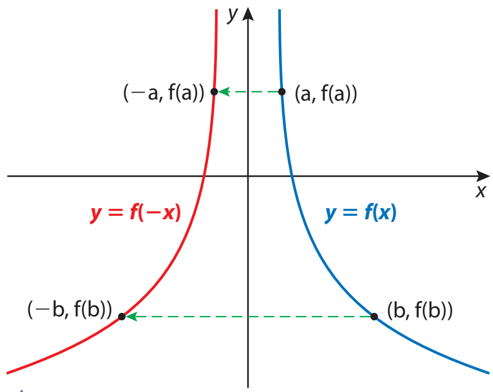



**Figure 2.23**

Figures 2.22 and 2.23 illustrate that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the $y$-axis.

**Reflections of a function in the coordinate axes**

I. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the $x$-axis.
II. The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the $y$-axis.

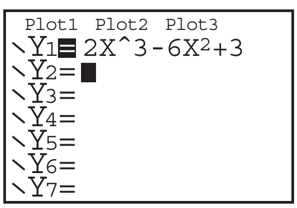
## Example 20

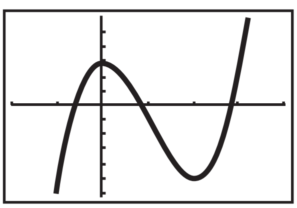
For $g(x) = 2x^3 - 6x^2 + 3$, find:

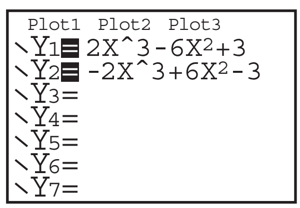
a) the function $h(x)$ that is the reflection of $g(x)$ in the $x$-axis
b) the function $p(x)$ that is the reflection of $g(x)$ in the $y$-axis.

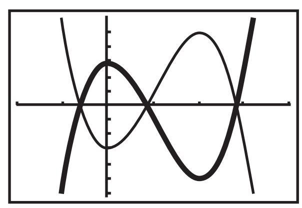
### *Solution*

a) Knowing that $y = -f(x)$ is the reflection of $y = f(x)$ in the $x$-axis, then $h(x) = -g(x) = -(2x^3 - 6x^2 + 3) \Rightarrow h(x) = -2x^3 + 6x^2 - 3$ will be the reflection of $g(x)$ in the $x$-axis. We can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.

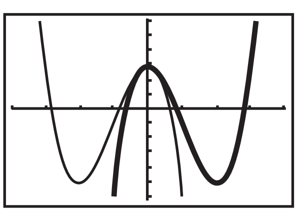


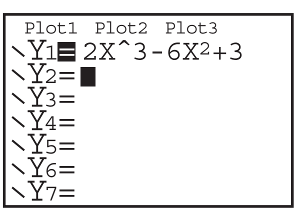



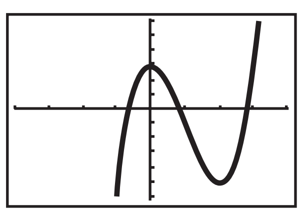


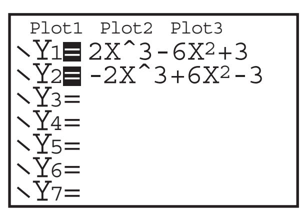



b) Knowing that $y = f(-x)$ is the reflection of $y = f(x)$ in the $y$-axis, we need to substitute $-x$ in for $x$ in $y = g(x)$. Thus, $p(x) = g(-x) = 2(-x)^3 - 6(-x)^2 + 3 \Rightarrow p(x) = -2x^3 - 6x + 3$ will be the reflection of $g(x)$ in the $y$-axis. Again, we can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.





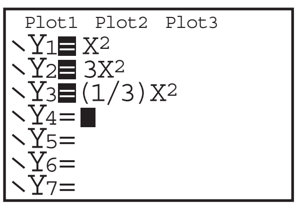


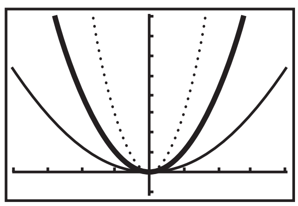
## Non-rigid transformations: stretching and shrinking

Horizontal and vertical translations, and reflections in the $x$- and $y$-axes are called **rigid transformations** because the shape of the graph does not change – only its position is changed. **Non-rigid transformations** cause the shape of the original graph to change. The non-rigid transformations that we will study cause the shape of a graph to *stretch* or *shrink* in either the vertical or horizontal direction.

### Vertical stretch or shrink

Graph the following three functions: $f(x) = x^2$, $g(x) = 3x^2$ and $h(x) = \frac{1}{3}x^2$. How do the graphs of $g$ and $h$ compare to the graph of $f$? Clearly, the shape of the graphs of $g$ and $h$ is not the same as the graph of $f$. Multiplying the function $f$ by a positive number greater than one, or less than one, has distorted the shape of the graph. For a certain value of $x$, the $y$-coordinate of $y = 3x^2$ is three times the $y$-coordinate of $y = x^2$. Therefore, the graph of $y = 3x^2$ can be obtained by *vertically shrinking* the graph of $y = x^2$ by a factor of 3 (**scale factor 3**). Likewise, the graph of $y = \frac{1}{3}x^2$ can be obtained by *vertically shrinking* the graph of $y = x^2$ by **scale factor** $\frac{1}{3}$.



Figures 2.24 and 2.25 illustrate how multiplying a function by a positive number, $a$, *greater than one* causes a transformation by which the function *stretches* vertically by scale factor $a$. A point $(x, y)$ on the graph of $y = f(x)$ is transformed to the point $(x, ay)$ on the graph of $y = af(x)$.

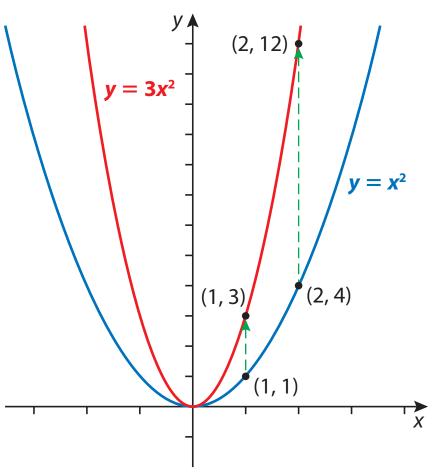



Figure 2.24

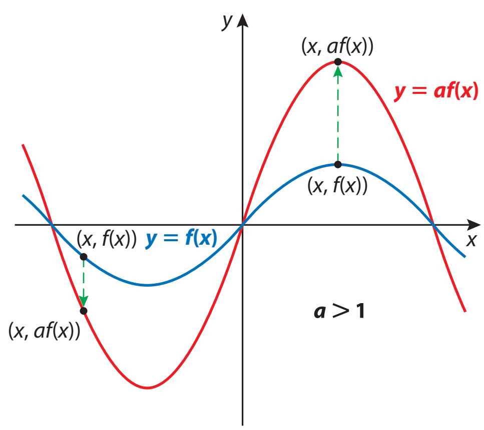



Figure 2.25

Figures 2.26 and 2.27 illustrate how multiplying a function by a positive number, $a$, *greater than zero and less than one* causes the function to *shrink* vertically by scale factor $a$. A point $(x, y)$ on the graph of $y = f(x)$ is transformed to the point $(x, ay)$ on the graph of $y = af(x)$.

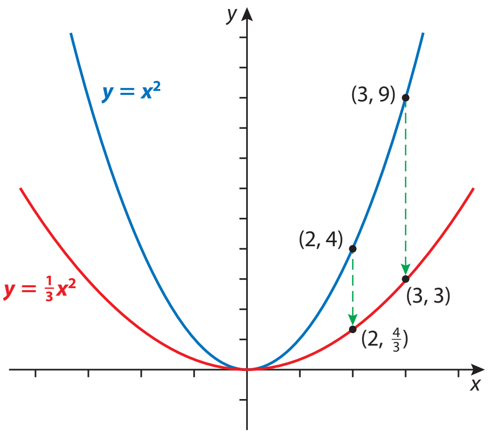



Figure 2.26

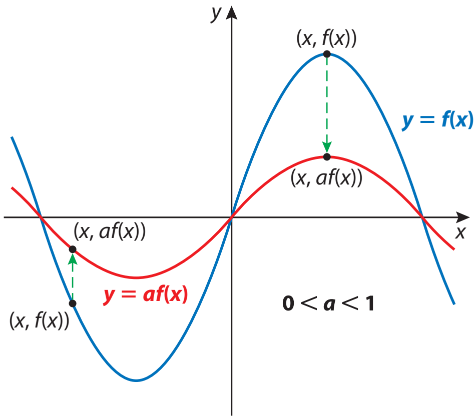



Figure 2.27

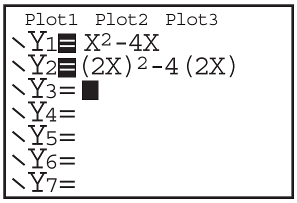
**Vertical stretching and shrinking of functions**

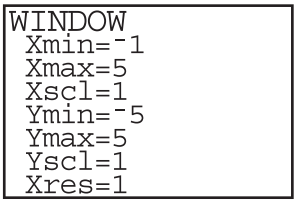
I. If $a > 1$, the graph of $y = af(x)$ is obtained by *vertically stretching* the graph of $y = f(x)$.

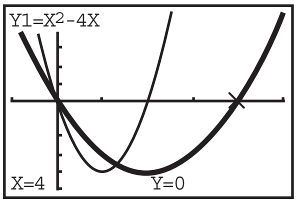
II. If $0 < a < 1$, the graph of $y = af(x)$ is obtained by *vertically shrinking* the graph of $y = f(x)$.

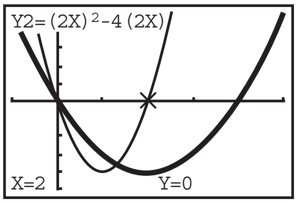
## Horizontal stretch or shrink

Let's investigate how the graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$. Given $f(x) = x^2 - 4x$, find another function, $g(x)$, such that $g(x) = f(2x)$. We substitute $2x$ for $x$ in the function $f$, giving $g(x) = (2x)^2 - 4(2x)$. For the purposes of our investigation, let's leave $g(x)$ in this form. On your GDC, graph these two functions, $f(x) = x^2 - 4x$ and $g(x) = (2x)^2 - 4(2x)$, using the indicated viewing window and graphing $f$ in bold style.




Comparing the graphs of the two equations, we see that $y = g(x)$ is *not* a translation or a reflection of $y = f(x)$. It is similar to the *shrinking* effect that occurs for $y = af(x)$ when $0 < a < 1$, except, instead of a vertical shrinking, the graph of $y = g(x) = f(2x)$ is obtained by *horizontally* shrinking the graph of $y = f(x)$. Given that it is a shrinking – rather than a stretching – the scale factor must be less than one. Consider the point $(4, 0)$ on the graph of $y = f(x)$. The point on the graph of $y = g(x) = f(2x)$ with the same $y$-coordinate and on the same side of the parabola is (2, 0). The $x$-coordinate of the point on $y = f(2x)$ is the $x$-coordinate of the point on $y = f(x)$ multiplied by $\frac{1}{2}$. Use your GDC to confirm this for other pairs of corresponding points on $y = x^2 - 4x$ and $y = (2x)^2 - 4(2x)$ that have the same $y$-coordinate. The graph of $y = f(2x)$ can be obtained by *horizontally shrinking* the graph of $y = f(x)$ with scale factor $\frac{1}{2}$. This makes sense because if $f(2x_2) = (2x_2)^2 - 4(2x_2)$ and $f(x_1) = x_1^2 - 4x_1$ are to produce the same $y$-value then $2x_2 = x_1$; and, thus $x_2 = \frac{1}{2}x_1$. Figures 2.28 and 2.29 illustrate how multiplying the $x$ variable of a function by a positive number, $a$, *greater than one* causes the function to *shrink* horizontally by scale factor $\frac{1}{a}$. A point $(x, y)$ on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

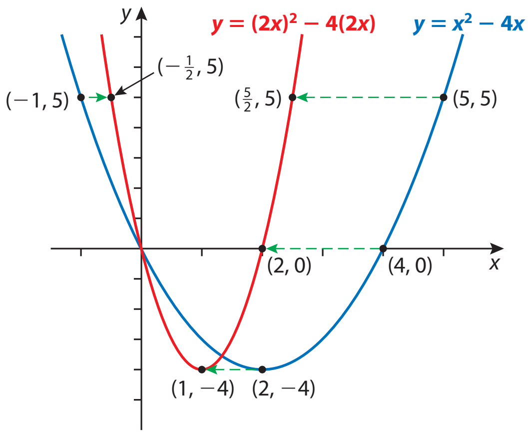



**Figure 2.28**

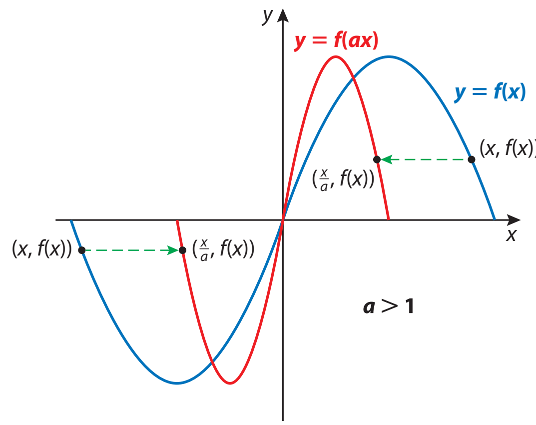
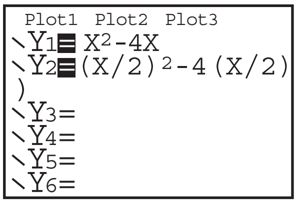
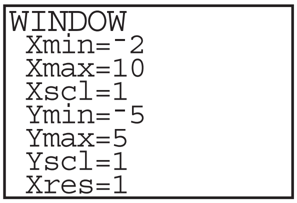
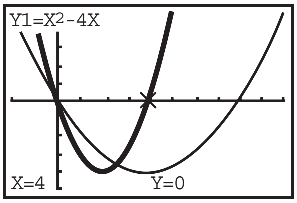


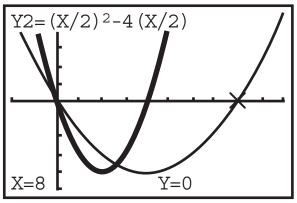

**Figure 2.29**

If $0 < a < 1$, the graph of the function $y = f(ax)$ is obtained by a *horizontal stretching* of the graph of $y = f(x)$ – rather than a shrinking – because the scale factor $\frac{1}{a}$ will be a value greater than 1 if $0 < a < 1$. Now, letting $a = \frac{1}{2}$ and, again using the function $f(x) = x^2 - 4x$, find $g(x)$, such that $g(x) = f\left(\frac{1}{2}x\right)$. We substitute $\frac{x}{2}$ for $x$ in $f$, giving $g(x) = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$. On your GDC, graph the functions $f$ and $g$ using the indicated viewing window with $f$ in bold.




The graph of $y = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$ is a horizontal stretching of the graph of $y = x^2 - 4x$ by scale factor $\frac{1}{a} = \frac{1}{\frac{1}{2}} = 2$. For example, the point (4, 0) on $y = f(x)$ has been moved horizontally to the point (8, 0) on $y = g(x) = f\left(\frac{x}{2}\right)$.

Figures 2.30 and 2.31 illustrate how multiplying the $x$ variable of a function by a positive number, $a$, *greater than zero and less than one* causes the function to *stretch* horizontally by scale factor $\frac{1}{a}$. A point $(x, y)$ on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

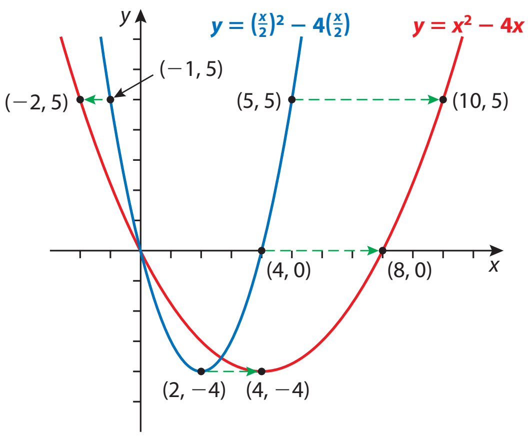



Figure 2.30

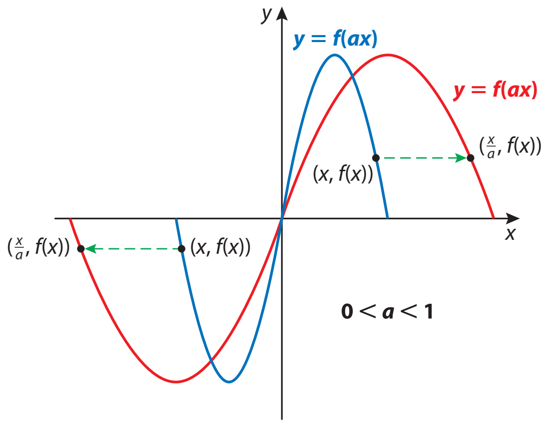



Figure 2.31

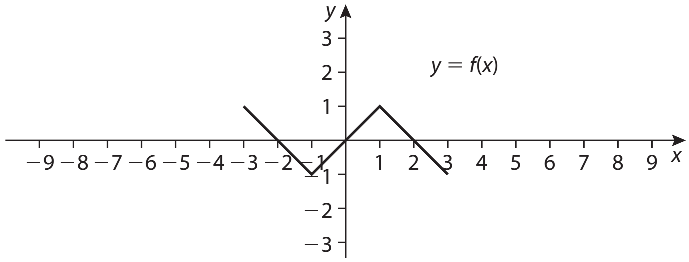
**Horizontal stretching and shrinking of functions**

I. If $a > 1$, the graph of $y = f(ax)$ is obtained by *horizontally shrinking* the graph of $y = f(x)$.

II. If $0 < a < 1$, the graph of $y = f(ax)$ is obtained by *horizontally stretching* the graph of $y = f(x)$.

## Example 21

The graph of $y = f(x)$ is shown at right. Sketch the graph of each of the following two functions.



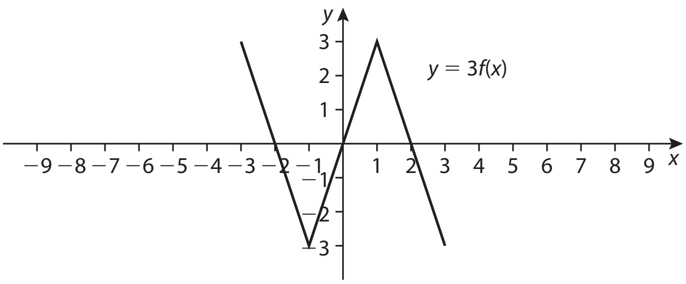

a) $y = 3f(x)$

b) $y = \frac{1}{3}f(x)$

c) $y = f(3x)$

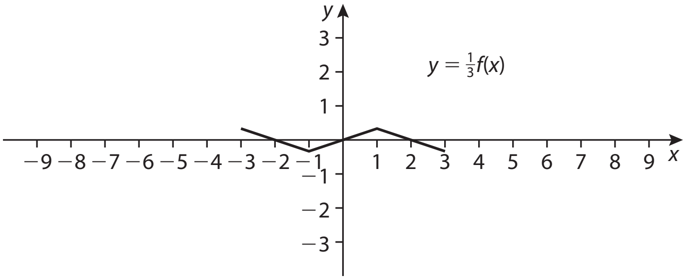
d) $y = f\left(\frac{1}{3}x\right)$

### *Solution*

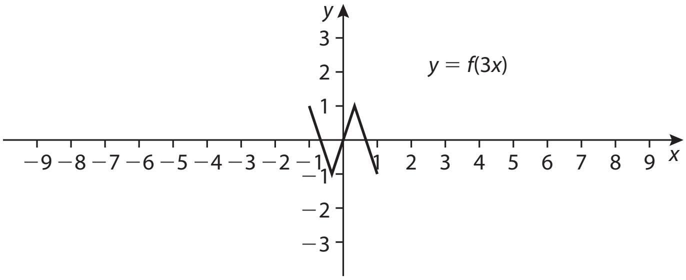



a) The graph of $y = 3f(x)$ is obtained by vertically stretching the graph of $y = f(x)$ with scale factor 3.

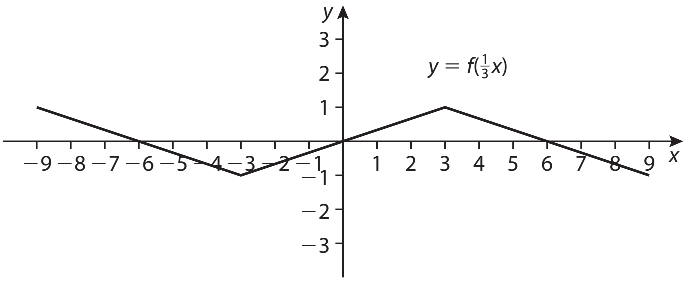
b) The graph of $y = \frac{1}{3}f(x)$ is obtained by vertically shrinking the graph of $y = f(x)$ with scale factor $\frac{1}{3}$.




c) The graph of $y = f(3x)$ is obtained by horizontally shrinking the graph of $y = f(x)$ with scale factor $\frac{1}{3}$.




d) The graph of $y = f\left(\frac{1}{3}x\right)$ is obtained by horizontally stretching the graph of $y = f(x)$ with scale factor 3.




**Example 22**

Describe the sequence of transformations performed on the graph of $y = x^2$ to obtain the graph of $y = 4x^2 - 3$.

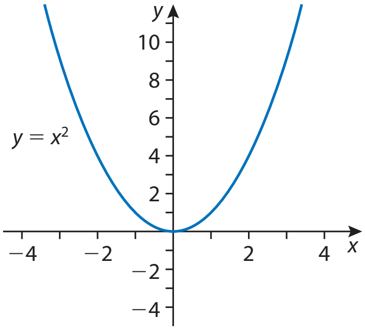
***Solution***

Step 1: Start with the graph of $y = x^2$.

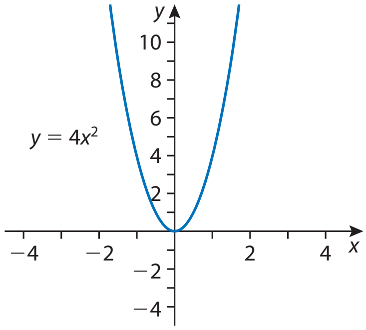
Step 2: Vertically stretch $y = x^2$ by scale factor 4.

Step 3: Vertically translate $y = 4x^2$ three units down.

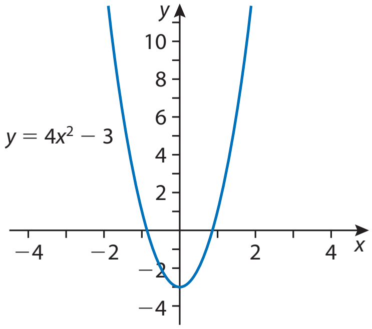
Step1:




Step2:




Step3:



Note that in Example 22, a vertical stretch followed by a vertical translation does not produce the same graph if the two transformations are performed in reverse order. A vertical translation followed by a vertical stretch would generate the following sequence of equations:

Step1: $y = x^2$ Step 2: $y = x^2 - 3$ Step 3: $y = 4(x^2 - 3) = 4x^2 - 12$

This final equation is not the same as $y = 4x^2 - 3$.

When combining two or more transformations, the order in which they are performed can make a difference. In general, when a sequence of transformations includes a vertical/horizontal stretch or shrink, or a reflection through the $x$-axis, the order may make a difference.

**Summary of transformations on the graphs of functions**

Assume that $a$, $h$ and $k$ are positive real numbers.

| Transformed function | Transformation performed on $y = f(x)$ |
|---|---|
| $y = f(x) + k$ | vertical translation $k$ units up |
| $y = f(x) - k$ | vertical translation $k$ units down |
| $y = f(x - h)$ | horizontal translation $h$ units right |
| $y = f(x + h)$ | horizontal translation $h$ units left |
| $y = -f(x)$ | reflection in the $x$-axis |
| $y = f(-x)$ | reflection in the $y$-axis |
| $y = af(x)$ | vertical stretch ($a > 1$) or shrink ($0 < a < 1$) |
| $y = f(ax)$ | horizontal stretch ($0 < a < 1$) or shrink ($a > 1$) |

## Exercise 2.4

In questions 1–8, sketch the graph of $f$, without a GDC or plotting points, by using your knowledge of some of the basic functions shown in Figure 2.15.

**1** $f: x \mapsto x^2 - 6$

**2** $f: x \mapsto (x - 6)^2$

**3** $f: x \mapsto |x| + 4$

**4** $f: x \mapsto |x + 4|$

**5** $f: x \mapsto 5 + \sqrt{x - 2}$

**6** $f: x \mapsto \frac{1}{x - 3}$

**7** $f: x \mapsto \frac{1}{(x + 5)^2} + 2$

**8** $f: x \mapsto -x^3 - 4$

**9** $f: x \mapsto -|x - 1| + 6$

**10** $f: x \mapsto \sqrt{-x + 3}$

**11** $f: x \mapsto 3\sqrt{x}$

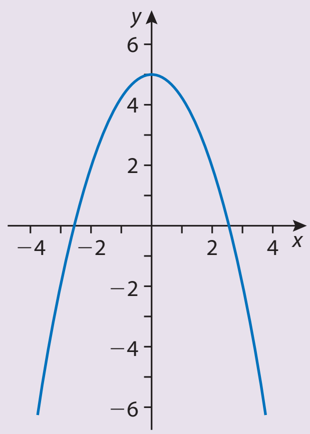
**12** $f: x \mapsto \frac{1}{2}x^2$

**13** $f: x \mapsto \left(\frac{1}{2}x\right)^2$

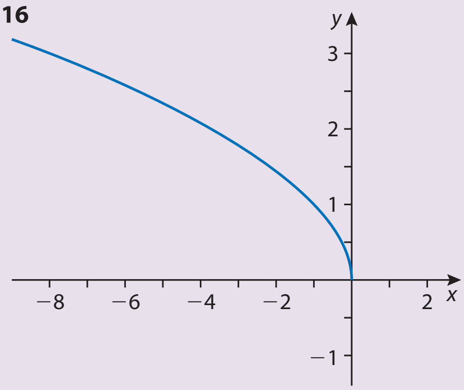
**14** $f: x \mapsto (-x)^3$

In questions 15–18, write the equation for the graph that is shown.

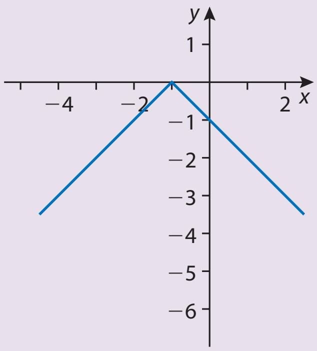
**15**



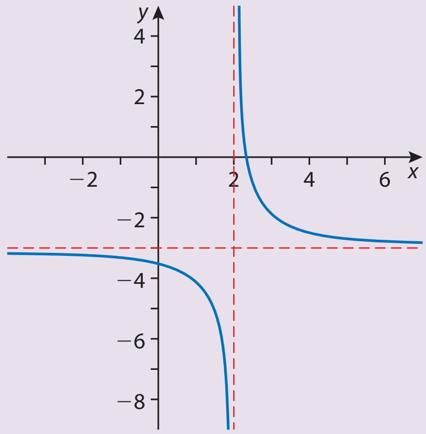

**16**



**17**

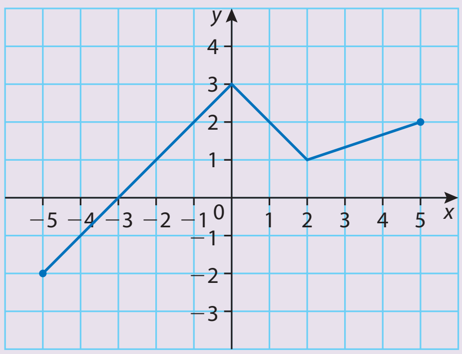



**18** Vertical and horizontal asymptotes shown




**19** The graph of $f$ is given. Sketch the graphs of the following functions.

a) $y = f(x) - 3$
b) $y = f(x - 3)$
c) $y = 2f(x)$
d) $y = f(2x)$
e) $y = -f(x)$
f) $y = f(-x)$
g) $y = 2(f(x) + 4)$
h) $y = 2f(x) + 4$




In questions 20–23, specify a sequence of transformations to perform on the graph of $y = x^2$ to obtain the graph of the given function.

**20** $g: x \mapsto (x - 3)^2 + 5$

**21** $h: x \mapsto -x^2 + 2$

**22** $p: x \mapsto \frac{1}{2}(x + 4)^2$

**23** $f: x \mapsto [3(x - 1)]^2 - 6$

In questions 24–26, a) express the quadratic function in the form $f(x) = a(x - h)^2 + k$, and b) state the coordinates of the vertex of the parabola with equation $y = f(x)$.

**24** $f(x) = x^2 + 6x + 2$

**25** $f(x) = x^2 - 2x + 4$

**26** $f(x) = 4x^2 - 4x - 1$

## 2.5 Quadratic functions

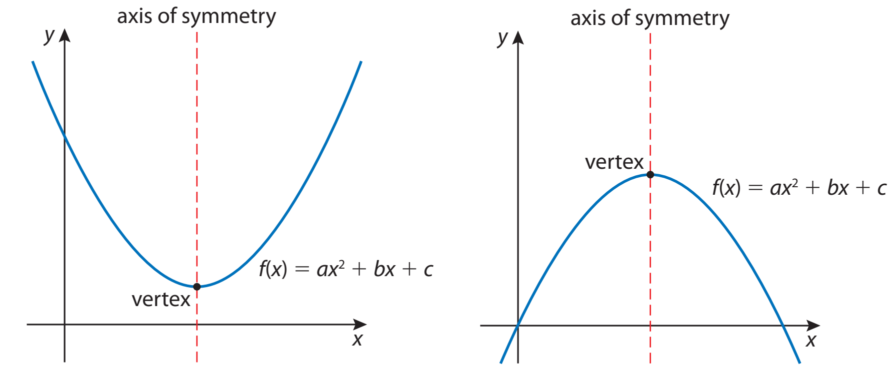
A **linear function** is a polynomial function of degree one that can be written in the general form $f(x) = ax + b$, where $a \neq 0$. The **degree** of a polynomial written in terms of $x$ refers to the largest exponent for $x$ in any terms of the polynomial. In this section, we will consider **quadratic functions** that are second degree polynomial functions often written in the general form $f(x) = ax^2 + bx + c$. Examples of quadratic functions, such as $f(x) = x^2 + 2$ (where $a = 1$, $b = 0$ and $c = 2$) and $f(x) = x^2 - 4x$ (where $a = 1$, $b = -4$ and $c = 0$), appeared earlier in this chapter.

The word *quadratic* comes from the Latin word *quadratus* that means four-sided, to make square, or simply a square. *Numerus quadratus* means a square number. Before modern algebraic notation was developed in the 17th and 18th centuries, the geometric figure of a square was used to indicate a number multiplying itself. Hence, raising a number to the power of two (in modern notation) is commonly referred to as the operation of squaring. *Quadratic* then came to be associated with a polynomial of degree two rather than being associated with the number four, as the prefix *quad* often indicates (e.g. quadruple).

**Definition of a quadratic function**

If $a$, $b$ and $c$ are real numbers, and $a \neq 0$, the function $f(x) = ax^2 + bx + c$ is a **quadratic function**. The graph of $f$ is the graph of the equation $y = ax^2 + bx + c$ and is called a **parabola**.

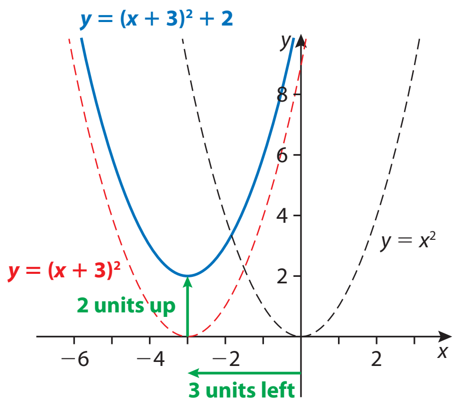
**Figure 2.32**




If $a > 0$ then the parabola opens upward

If $a < 0$ then the parabola opens downward

**Figure 2.33**



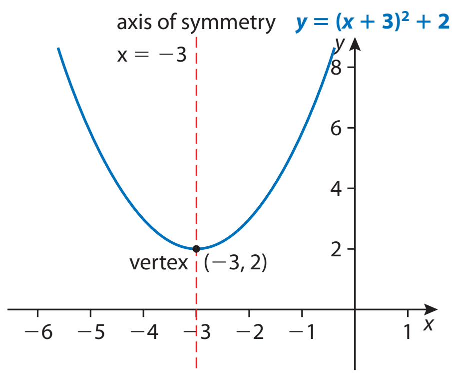




**Figure 2.34**

Each parabola is symmetric about a vertical line called its **axis of symmetry.** The axis of symmetry passes through a point on the parabola called the **vertex** of the parabola, as shown in Figure 2.32. If the leading coefficient, $a$, of the quadratic function $f(x) = ax^2 + bx + c$ is positive, the parabola opens upward (concave up) – and the $y$-coordinate of the vertex will be a **minimum value** for the function. If the leading coefficient, $a$, of $f(x) = ax^2 + bx + c$ is negative, the parabola opens downward (concave down) – and the $y$-coordinate of the vertex will be a **maximum value** for the function.

## The graph of $f(x) = a(x - h)^2 + k$

From the previous section, we know that the graph of the equation $y = (x + 3)^2 + 2$ can be obtained by translating $y = x^2$ three units to the left and two units up. Being familiar with the shape and position of the graph of $y = x^2$ and knowing the two translations that transform $y = x^2$ to $y = (x + 3)^2 + 2$, we can easily visualize and/or sketch the graph of $y = (x + 3)^2 + 2$ (see Figure 2.33). We can also determine the axis of symmetry and the vertex of the graph. Figure 2.34 shows that the graph of $y = (x + 3)^2 + 2$ has an axis of symmetry of $x = -3$ and a vertex at $(-3, 2)$. The equation $y = (x + 3)^2 + 2$ can also be written as $y = x^2 + 6x + 11$. Because we can easily identify the vertex of the parabola when the equation is written as $y = (x + 3)^2 + 2$, we often refer to this as the **vertex form** of the quadratic equation, and $y = x^2 + 6x + 11$ as the **general form**.

- **Hint:** $f(x) = a(x - h)^2 + k$ is sometimes referred to as the **standard form** of a quadratic function.

**Vertex form of a quadratic function**

If a quadratic function is written in the form $f(x) = a(x - h)^2 + k$, with $a \neq 0$, the graph of $f$ has an axis of symmetry of $x = h$ and a vertex at $(h, k)$.

## Completing the square

For visualizing and sketching purposes, it is helpful to have a quadratic function written in vertex form. How do we rewrite a quadratic function written in the form $f(x) = ax^2 + bx + c$ (general form) into the form $f(x) = a(x - h)^2 + k$ (vertex form)? We use the technique of **completing the square.**

For any real number $p$, the quadratic expression $x^2 + px + \left(\frac{p}{2}\right)^2$ is the square of $\left(x + \frac{p}{2}\right)$. Convince yourself of this by expanding $\left(x + \frac{p}{2}\right)^2$. The technique of *completing the square* is essentially the process of adding a constant to a quadratic expression to make it the square of a binomial. If the coefficient of the quadratic term ($x^2$) is a positive one, the coefficient of the linear term is $p$, and the constant term is $\left(\frac{p}{2}\right)^2$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ and the square is completed.

Remember that the coefficient of the quadratic term (leading coefficient) must be equal to positive one before completing the square.

### Example 23

Find the equation of the axis of symmetry and the coordinates of the vertex of the graph of $f(x) = x^2 - 8x + 18$ by rewriting the function in the form $f(x) = a(x - h)^2 + k$.

#### *Solution*

To complete the square and get the quadratic expression $x^2 - 8x + 18$ in the form $x^2 + px + \left(\frac{p}{2}\right)^2$, the constant term needs to be $\left(\frac{-8}{2}\right)^2 = 16$. We need to add 16, but also subtract 16, so that we are adding zero overall and, hence, not changing the original expression.

$f(x) = x^2 - 8x + 16 - 16 + 18$ actually adding zero $(-16 + 16)$ to the right side

$f(x) = x^2 - 8x + 16 + 2$ $x^2 - 8x + 16$ fits the pattern $x^2 + px + \left(\frac{p}{2}\right)^2$ with $p = -8$

$f(x) = (x - 4)^2 + 2$ $x^2 - 8x + 16 = (x - 4)^2$

The axis of symmetry of the graph of $f$ is the vertical line $x = 4$ and the vertex is at (4, 2). See Figure 2.35.

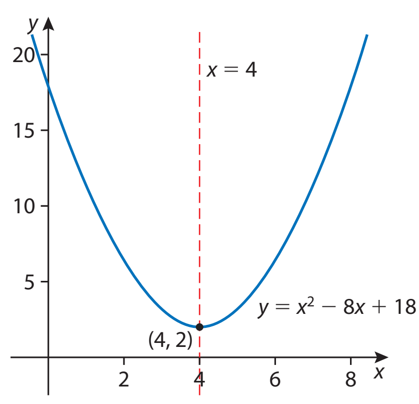



Figure 2.35

### Example 24

For the function $g: x \mapsto -2x^2 - 12x + 7$,

a) find the axis of symmetry and the vertex of the graph

b) indicate the transformations that can be applied to $y = x^2$ to obtain the graph

c) find the minimum or maximum value.

***Solution***

a) $g: x \mapsto -2\left(x^2 + 6x - \frac{7}{2}\right)$ — factorize so that the coefficient of the quadratic term is $+1$

$g: x \mapsto -2\left(x^2 + 6x + 9 - 9 - \frac{7}{2}\right)$ — $p = 6 \Rightarrow \left(\frac{p}{2}\right)^2 = 9$; hence, add $+9\ -9$ (zero)

$g: x \mapsto -2\left[(x + 3)^2 - \frac{18}{2} - \frac{7}{2}\right]$ — $x^2 + 6x + 9 = (x + 3)^2$

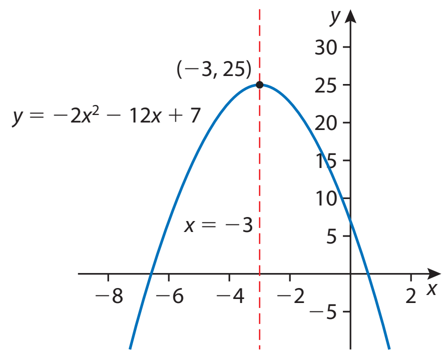
$g: x \mapsto -2\left[(x + 3)^2 - \frac{25}{2}\right]$

$g: x \mapsto -2(x + 3)^2 + 25$ — multiply through by $-2$ to remove outer brackets

$g: x \mapsto -2(x - (-3))^2 + 25$ — express in vertex form $g: x \mapsto a(x - h)^2 + k$

The axis of symmetry of the graph of $g$ is the vertical line $x = -3$ and the vertex is at $(-3, 25)$. See Figure 2.36.

**Figure 2.36**




b) Since $g: x \mapsto -2x^2 - 12x + 7 = -2(x + 3)^2 + 25$, the graph of $g$ can be obtained by applying the following transformations (in the order given) on the graph of $y = x^2$: horizontal translation of 3 units left; reflection in the $x$-axis (parabola opening down); vertical stretch of factor 2; and a vertical translation of 25 units up.

c) The parabola opens down because the leading coefficient is negative. Therefore, $g$ has a maximum and no minimum value. The maximum value is 25 ($y$-coordinate of vertex) at $x = -3$.

The technique of completing the square can be used to derive the quadratic formula. The following example derives a general expression for the axis of symmetry and vertex of a quadratic function in the general form $f(x) = ax^2 + bx + c$ by completing the square.

**Example 25**

Find the axis of symmetry and the vertex for the general quadratic function $f(x) = ax^2 + bx + c$.

***Solution***

$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ factorize so that the coefficient of the $x^2$ term is $+1$

$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ $p = \frac{b}{a} \Rightarrow \left(\frac{p}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$ $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ multiply through by $a$

$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$ express in vertex form $f(x) = a(x - h)^2 + k$

This result leads to the following generalization.

**Symmetry and vertex of $f(x) = ax^2 + bx + c$**

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the vertical line with the equation $x = -\frac{b}{2a}$ and the vertex has coordinates $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Check the results for Example 24 using the formulae for the axis of symmetry and vertex. For the function $g: x \mapsto -2x^2 - 12x + 7$:

$x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3 \Rightarrow$ axis of symmetry is the vertical line $x = -3$

$c - \frac{b^2}{4a} = 7 - \frac{(-12)^2}{4(-2)} = \frac{56}{8} + \frac{144}{8} = 25 \Rightarrow$ vertex has coordinates $(-3, 25)$

These results agree with the results from Example 24.

## Zeros of a quadratic function

A specific value for $x$ is a **zero** (or **root**) of a quadratic function $f(x) = ax^2 + bx + c$ if it is a solution to the equation $ax^2 + bx + c = 0$. For this course, we are only concerned with values of $x$ that are real numbers. The $x$-coordinate of any point(s) where $f$ crosses the $x$-axis ($y$-coordinate is zero) is a zero of the function. A quadratic function can have no, one or two real zeros as Table 2.3 illustrates. Finding the zeros of a quadratic function requires you to solve quadratic equations of the form $ax^2 + bx + c = 0$. Although $a \neq 0$, it is possible for $b$ or $c$ to be equal to zero. There are five general methods for solving quadratic equations as outlined in the table on page 70.

| | |
|---|---|
| **Square root** | If $a^2 = c$ and $c > 0$, then $a = \pm\sqrt{c}$. |
| **Examples** | $x^2 - 25 = 0$ <br> $x^2 = 25$ <br> $x = \pm\sqrt{5}$ <br><br> $(x + 2)^2 = 15$ <br> $x + 2 = \pm\sqrt{15}$ <br> $x = -2 \pm\sqrt{15}$ |
| **Factorizing** | If $ab = 0$, then $a = 0$ or $b = 0$. |
| **Examples** | $x^2 + 3x - 10 = 0$ <br> $(x + 5)(x - 2) = 0$ <br> $x = -5$ or $x = 2$ <br><br> $x^2 - 7x = 0$ <br> $x(x - 7) = 0$ <br> $x = 0$ or $x = 7$ |
| **Completing the square** | If $x^2 + px + q = 0$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$ that leads to $\left(x + \frac{p}{2}\right)^2 = -q + \frac{p^2}{4} \ldots$ <br> and then the square root of both sides (as above). |
| **Example** | $x^2 - 8x + 5 = 0$ <br> $x^2 - 8x + 16 = -5 + 16$ <br> $(x - 4)^2 = 11$ <br> $x - 4 = \pm\sqrt{11}$ <br> $x = 4 \pm \sqrt{11}$ |
| **Quadratic formula** | If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. |
| **Example** | $2x^2 - 3x - 4 = 0$ <br> $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$ <br> $x = \frac{3 \pm \sqrt{41}}{4}$ |
| **Graphing** | Graph the equation $y = ax^2 + bx + c$ on your GDC. Use the calculating features of your GDC to determine the <br> $x$-coordinates of the point(s) where the parabola intersects the $x$-axis. |
| **Example** | $2x^2 - 5x - 7 = 0$ GDC calculations reveal that the zeros are $x = \frac{7}{2}$ and $x = -1$ |

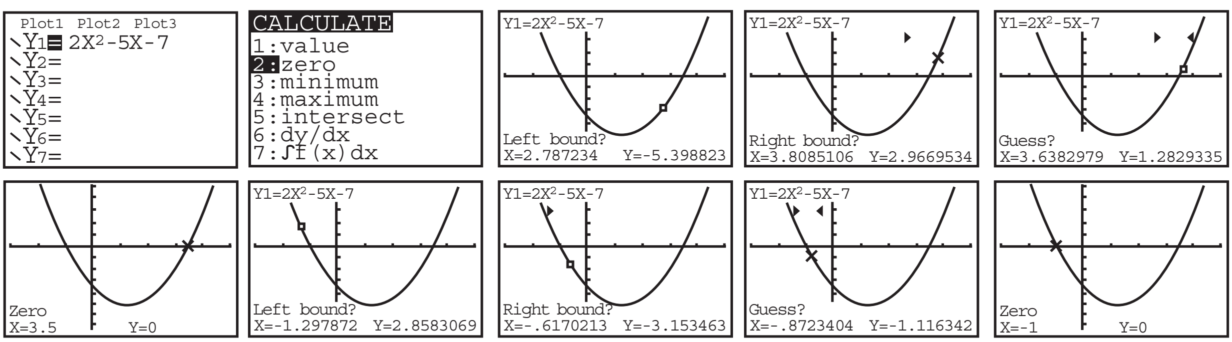



**Table 2.3** Methods for solving quadratic equations.

## The quadratic formula and the discriminant

The expression $b^2 - 4ac$ in the quadratic formula has special significance because you need to take the positive and negative square root of $b^2 - 4ac$ when using the quadratic formula. Hence, whether $b^2 - 4ac$ (often labelled $\Delta$; read 'delta') is positive, negative or zero will determine the number of real solutions for the quadratic equation $ax^2 + bx + c = 0$, and, consequently, also the number of times the graph of $f(x) = ax^2 + bx + c$ intersects the $x$-axis ($y = 0$).

**For the quadratic function $f(x) = ax^2 + bx + c$, ($a \neq 0$):**

If $\Delta = b^2 - 4ac > 0$, $f$ has two distinct real solutions, and the graph of $f$ intersects the $x$-axis twice.

If $\Delta = b^2 - 4ac = 0$, $f$ has one real solution (a double root), and the graph of $f$ intersects the $x$-axis once (i.e. it is a tangent to the $x$-axis).

If $\Delta = b^2 - 4ac < 0$, $f$ has no real solutions, and the graph of $f$ does not intersect the $x$-axis.

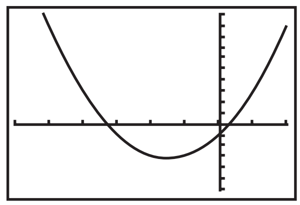
## Example 26

Use the discriminant to determine how many real solutions each equation has. Visually confirm the result by graphing the corresponding quadratic function for each equation on your GDC.

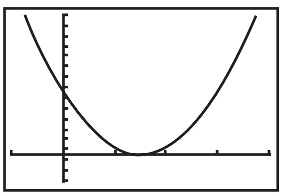
a) $x^2 + 3x - 1 = 0$  b) $4x^2 - 12x + 9 = 0$  c) $2x^2 - 5x + 6 = 0$

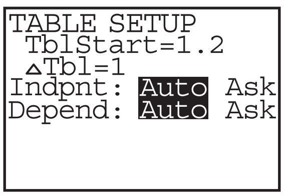
### Solution

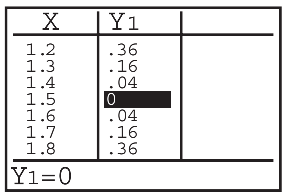
a) The discriminant is $\Delta = 3^2 - 4(1)(-1) = 13 > 0$. Therefore, the equation has two distinct real zeros. This result is confirmed by the graph of the quadratic function $y = x^2 + 3x - 1$ that clearly shows it intersecting the $x$-axis twice.



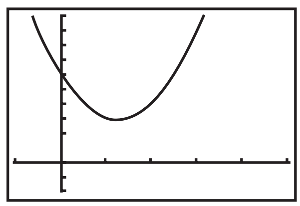
b) The discriminant is $\Delta = (-12)^2 - 4(4)(9) = 0$. Therefore, the equation has one real zero. The graph on the GDC of $y = 4x^2 - 12x + 9$ appears to intersect the $x$-axis at only one point. We can be more confident with this conclusion by investigating further – for example, tracing or looking at a table of values on the GDC.






c) The discriminant is $\Delta = (-5)^2 - 4(2)(6) = -23 < 0$. Therefore, the equation has no real zeros. This result is confirmed by the graph of the quadratic function $y = 2x^2 - 5x + 6$ that clearly shows that the graph does not intersect the $x$-axis.



## Example 27

For $4x^2 + 4kx + 9 = 0$, determine the value(s) of $k$ so that the equation has: a) one real zero, b) two distinct real zeros, and c) no real zeros.

### Solution

a) For one real zero:
$\Delta = (4k)^2 - 4(4)(9) = 0 \Rightarrow 16k^2 - 144 = 0 \Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

b) For two distinct real zeros: $\Delta = (4k)^2 - 4(4)(9) > 0 \Rightarrow 16k^2 > 144 \Rightarrow k^2 > 9 \Rightarrow k < -3$ or $k > 3$

c) For no real zeros: $\Delta = (4k)^2 - 4(4)(9) < 0 \Rightarrow 16k^2 < 144 \Rightarrow k^2 < 9 \Rightarrow k > -3$ and $k < 3 \Rightarrow -3 < k < 3$

## The graph of $f(x) = a(x - p)(x - q)$

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$ then we can easily identify the $x$-intercepts of the graph of $f$. Consider that $f(p) = a(p - p)(p - q) = a(0)(p - q) = 0$ and that $f(q) = a(q - p)(q - q) = a(q - p)(0) = 0$. Therefore, the quadratic function $f(x) = a(x - p)(x - q)$ will intersect the $x$-axis at the points $(p, 0)$ and $(q, 0)$. We need to factorize in order to rewrite a quadratic function in the form $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - p)(x - q)$. Hence, $f(x) = a(x - p)(x - q)$ can be referred to as the **factorized** form of a quadratic function. Recalling the symmetric nature of a parabola, it is clear that the $x$-intercepts $(p, 0)$ and $(q, 0)$ will be equidistant from the axis of symmetry (see Figure 2.37). As a result, the equation of the axis of the symmetry and the $x$-coordinate of the vertex of the parabola can be found from finding the average of $p$ and $q$.

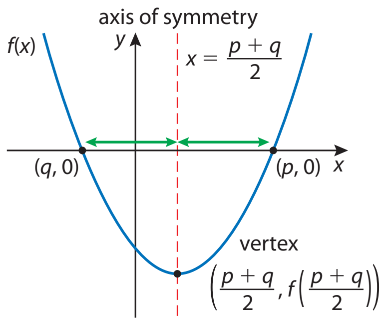



**Figure 2.37**

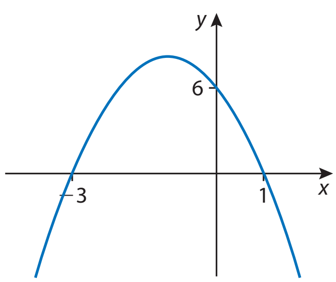
**Factorized form of a quadratic function**

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$, with $a \neq 0$, the graph of $f$ has $x$-intercepts at $(p, 0)$ and $(q, 0)$, an axis of symmetry with equation $x = \frac{p+q}{2}$, and a vertex at $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$.

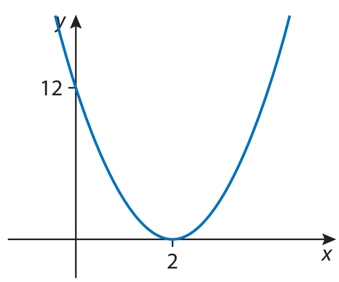
### Example 28

Find the equation of each quadratic function from the graph in the form $f(x) = a(x - p)(x - q)$ and also in the form $f(x) = ax^2 + bx + c$.

a)




b)




***Solution***

a) Since the $x$-intercepts are $-3$ and $1$ then $y = a(x + 3)(x - 1)$. The $y$-intercept is 6, so when $x = 0$, $y = 6$. Hence, $6 = a(0 + 3)(0 - 1) = -3a \Rightarrow a = -2$ ($a < 0$ agrees with the fact that the parabola is opening down). The function is $f(x) = -2(x + 3)(x - 1)$, and expanding to remove brackets reveals that the function can also be written as $f(x) = -2x^2 - 4x + 6$.

b) The function has one $x$-intercept at 2 (double root), so $p = q = 2$ and $y = a(x - 2)(x - 2) = a(x - 2)^2$. The $y$-intercept is 12, so when $x = 0$, $y = 12$. Hence, $12 = a(0 - 2)^2 = 4a \Rightarrow a = 3$ ($a > 0$ agrees with the parabola opening up). The function is $f(x) = 3(x - 2)^2$. Expanding reveals that the function can also be written as $f(x) = 3x^2 - 12x + 12$.

### Example 29

The graph of a quadratic function intersects the $x$-axis at the points $(-6, 0)$ and $(-2, 0)$ and also passes through the point $(2, 16)$. a) Write the function in the form $f(x) = a(x - p)(x - q)$. b) Find the vertex of the parabola. c) Write the function in the form $f(x) = a(x - h)^2 + k$.

***Solution***

a) The $x$-intercepts of $-6$ and $-2$ gives $f(x) = a(x + 6)(x + 2)$. Since $f$ passes through $(2, 16)$, then $f(2) = 16 \Rightarrow f(2) = a(2 + 6)(2 + 2) = 16$ $\Rightarrow \quad 32a = 16 \quad \Rightarrow \quad a = \frac{1}{2}$. Therefore, $f(x) = \frac{1}{2}(x + 6)(x + 2)$.

b) The $x$-coordinate of the vertex is the average of the $x$-intercepts. $x = \dfrac{-6-2}{2} = -4$, then the $y$-coordinate of the vertex is $y = f(-4) = \frac{1}{2}(-4 + 6)(-4 + 2) = -2$. Hence, the vertex is $(-4, -2)$.

c) In vertex form, the quadratic function is $f(x) = \frac{1}{2}(x + 4)^2 - 2$.

### Exercise 2.5

For each of the quadratic functions $f$ in questions 1-5, find the following:

a) the equation for the axis of symmetry and the vertex by algebraic methods
b) the transformation(s) that can be applied to $y = x^2$ to obtain the graph of $y = f(x)$
c) the minimum or maximum value of $f$.

Check your results using your GDC.

**1** $f: x \mapsto x^2 - 10x + 32$

**2** $f: x \mapsto x^2 + 6x + 8$

**3** $f: x \mapsto -2x^2 - 4x + 10$

**4** $f: x \mapsto 4x^2 - 4x + 9$

**5** $f: x \mapsto \frac{1}{2}x^2 + 7x + 26$

In questions 6–13, solve the quadratic equation using factorization.

**6** $x^2 + 2x - 8 = 0$

**7** $x^2 = 3x + 10$

**8** $6x^2 - 9x = 0$

**9** $10 + 3x = x^2$

**10** $x^2 = 3x + 10$

**11** $3x^2 + 11x - 4 = 0$

**12** $3x^2 + 18 = 15x$

**13** $9x - 2 = 4x^2$

In questions 14–19, use the method of completing the square to solve the quadratic equation.

**14** $x^2 + 4x - 3 = 0$

**15** $x^2 - 4x - 5 = 0$

**16** $x^2 - 2x + 3 = 0$

**17** $2x^2 + 16x + 6 = 0$

**18** $x^2 + 2x - 8 = 0$

**19** $-2x^2 + 4x + 9 = 0$

**20** Let $f(x) = x^2 - 4x - 1$. a) Use the quadratic formula to find the zeros of the function. b) Use the zeros to find the equation for the axis of symmetry of the parabola. c) Find the minimum or maximum value of $f$.

In questions 21–24, determine the number of real solutions to each equation.

**21** $x^2 + 3x + 2 = 0$

**22** $2x^2 - 3x + 2 = 0$

**23** $x^2 - 1 = 0$

**24** $2x^2 - \frac{9}{4}x + 1 = 0$

**25** Find the value(s) of $p$ for which the equation $2x^2 + px + 1 = 0$ has one real solution.

**26** Find the value(s) of $m$ for which the equation $x^2 + 4x + k = 0$ has two distinct real solutions.

**27** The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions. Find the set of all possible values of $k$.

**28** Find all possible values of $m$ so that the graph of the function $g: x \mapsto mx^2 + 6x + m$ does not touch the $x$-axis.

## Practice questions

**1** The functions $f: x \mapsto \sqrt{x-3}$ and $g: x \mapsto x^2 + 2x$. The function $(f \circ g)(x)$ is defined for all $x \in \mathbb{R}$ **except** for the interval $]a, b[$.

**a)** Calculate the value of $a$ and of $b$.

**b)** Find the range of $f \circ g$.

**2** Two functions $g$ and $h$ are defined as $g(x) = 2x - 7$ and $h(x) = 3(2 - x)$.

Find: **a)** $g^{-1}(3)$

**b)** $(h \circ g)(6)$

**3** Consider the functions $f(x) = 5x - 2$ and $g(x) = \dfrac{4-x}{3}$.

**a)** Find $g^{-1}$.

**b)** Solve the equation $(f \circ g^{-1})(x) = 8$.

**4** The functions $g$ and $h$ are defined by $g: x \mapsto x - 3$ and $h: x \mapsto 2x$.

**a)** Find an expression for $(g \circ h)(x)$.

**b)** Show that $g^{-1}(14) + h^{-1}(14) = 24$.

**5** The function $f$ is defined by $f(x) = x^2 + 8x + 11$, for $x \geqslant -4$.

**a)** Write $f(x)$ in the form $(x - h)^2 + k$.

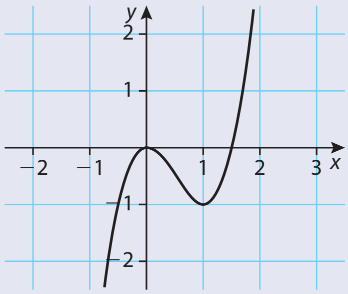
**b)** Find the inverse function $f^{-1}$.

**c)** State the domain of $f^{-1}$.

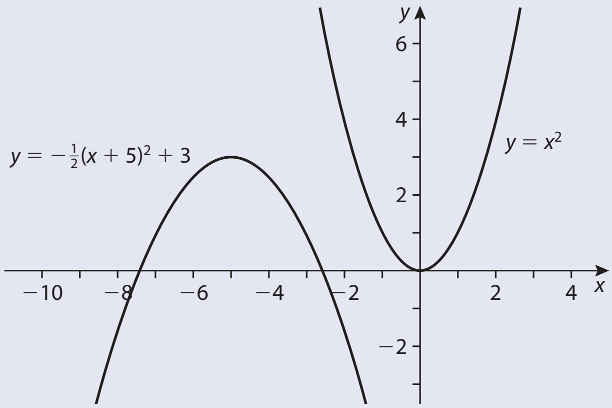
**6** The diagram right shows the graph of $y = f(x)$. It has maximum and minimum points at $(0, 0)$ and $(1, -1)$, respectively.

**a)** Copy the diagram, and on the same diagram draw the graph of $y = f(x + 1) - \frac{1}{2}$.

**b)** What are the coordinates of the minimum and maximum points of $y = f(x + 1) - \frac{1}{2}$?




**7** The diagram shows parts of the graphs of $y = x^2$ and $y = -\frac{1}{2}(x + 5)^2 + 3$.



The graph of $y = x^2$ may be transformed into the graph of $y = -\frac{1}{2}(x + 5)^2 + 3$ by these transformations.

A reflection in the line $y = 0$, followed by
a vertical stretch with scale factor $k$, followed by
a horizontal translation of $p$ units, followed by
a vertical translation of $q$ units.

Write down the value of

**a)** $k$ **b)** $p$ **c)** $q$.

**8** The function $f$ is defined by $f(x) = \frac{4}{\sqrt{16 - x^2}}$, for $-4 < x < 4$.

**a)** Without using a GDC, sketch the graph of $f$.

**b)** Write down the equation of each vertical asymptote.

**c)** Write down the range of the function $f$.

**9** Let $g: x \mapsto \frac{1}{x}$, $x \neq 0$.

**a)** Without using a GDC, sketch the graph of $g$.

The graph of $g$ is transformed to the graph of $h$ by a translation of 4 units to the left and 2 units down.

**b)** Find an expression for the function $h$.

**c)** **(i)** Find the $x$- and $y$-intercepts of $h$.
**(ii)** Write down the equations of the asymptotes of $h$.
**(iii)** Sketch the graph of $h$.

**10** Consider $f(x) = \sqrt{x + 3}$.

**a)** Find:
**(i)** $f(8)$
**(ii)** $f(46)$
**(iii)** $f(-3)$

**b)** Find the values of $x$ for which $f$ is undefined.

**c)** Let $g: x \mapsto x^2 - 5$. Find $(g \circ f)(x)$.

**11** Let $g(x) = \frac{x - 8}{2}$ and $h(x) = x^2 - 1$.

**a)** Find $g^{-1}(-2)$.

**b)** Find an expression for $(g^{-1} \circ h)(x)$.

**c)** Solve $(g^{-1} \circ h)(x) = 22$.

**12** Given the functions $f: x \mapsto 3x - 1$ and $g: x \mapsto \frac{4}{x}$, find the following:

**a)** $f^{-1}$ **b)** $f \circ g$ **c)** $(f \circ g)^{-1}$ **d)** $g \circ g$

**13** The quadratic function $f$ is defined by $f(x) = 2x^2 + 8x + 17$.

**a)** Write $f$ in the form $f(x) = 2(x - h)^2 + k$.

**b)** The graph of $f$ is translated 5 units in the positive $x$-direction and 2 units in the positive $y$-direction. Find the function $g$ for the translated graph, giving your answer in the form $g(x) = 2(x - h)^2 + k$.

**14** Let $g(x) = 3x^2 - 6x - 4$.

**a)** Express $g(x)$ in the form $g(x) = 3(x - h)^2 + k$.

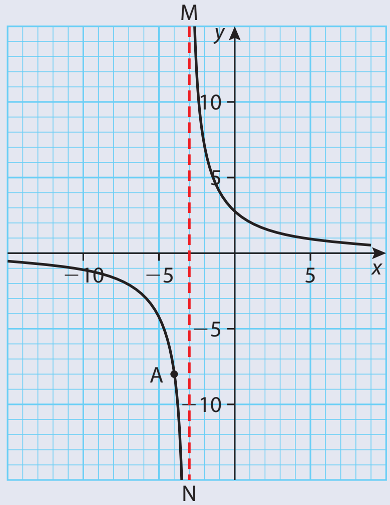
**b)** Write down the vertex of the graph of $g$.

**c)** Write down the equation of the axis of symmetry of the graph of $g$.

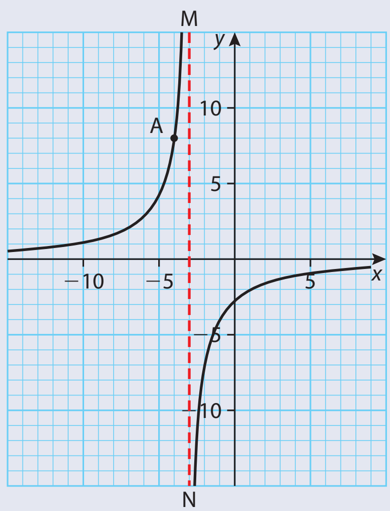
**d)** Find the $y$-intercept of the graph of $g$.

**e)** The $x$-intercepts of $g$ can be written as $\frac{p \pm \sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$. Find the value of $p$, $q$, and $r$.

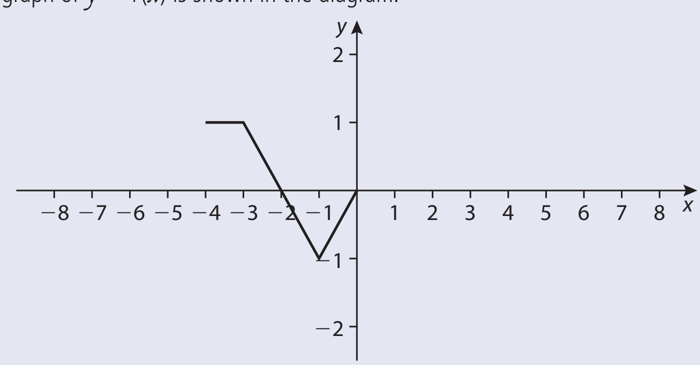
**15 a)** The diagram shows part of the graph of the function $h(x) = \frac{a}{x - b}$. The curve passes through the point A $(-4, -8)$. The vertical line (MN) is an asymptote. Find the value of: **(i)** $a$; **(ii)** $b$.




**b)** The graph of $h(x)$ is transformed as shown in the diagram right. The point A is transformed to A′$(-4, 8)$. Give a full geometric description of the transformation.

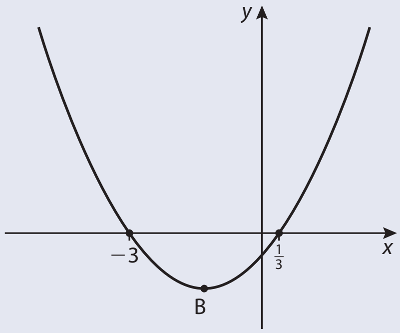



**16** The graph of $y = f(x)$ is shown in the diagram.



**a)** Make two copies of the coordinate system as shown in the above diagram but without the graph of $y = f(x)$. On the first diagram sketch a graph of $y = 2f(x)$, and on the second diagram sketch a graph of $y = f(x - 4)$.

**b)** The point A $(-3, 1)$ is on the graph of $y = f(x)$. The point A′ is the corresponding point on the graph of $y = -f(x) - 1$. Find the coordinates of A′.




**17** The diagram represents the graph of the function $f(x) = (x - p)(x - q)$.

**a)** Write down the values of $p$ and $q$.

**b)** The function has a minimum value at the point B. Find the x-coordinate of B.

**c)** Write the expression for $f(x)$ in the form $ax^2 + bx + c$.

**18** The diagram shows the parabola $y = (5 + x)(2 - x)$. The points A and C are the $x$-intercepts and the point B is the maximum point. Find the coordinates of A, B and C.