1 Trigonometry 1

Although most people connect trigonometry with the study of triangles, it is from the circle that this area of mathematics originates.

The study of trigonometry is not new. Its roots come from the Babylonians around 300 BC. This area of mathematics was further developed by the Ancient Greeks around 100 BC. Hipparchus, Ptolemy and Menelaus are considered to have founded trigonometry as we now know it. It was originally used to aid the study of astronomy.

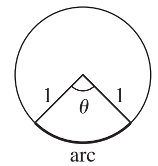
In the modern world trigonometry can be used to answer questions like "How far apart are each of the 32 pods on the London Eye?" and "What would a graph of someone's height on the London Eye look like?"

1.1 Circle problems

Radians

It is likely that up until now you have measured angles in degrees, but as for most measurements, there is more than one unit that can be used.

Consider a circle with radius 1 unit.



As θ increases, the arc length increases. For a particular value of θ, the arc will be the same length as the radius. When this occurs, the angle is defined to be 1 radian.

The circumference of a circle is given by $C = 2\pi r$, so when $r = 1$, $C = 2\pi$.

As there are 360° at the centre of a circle, and 1 radian is defined to be the angle subtended by an arc of length 1,

$$2\pi \text{ radians} = 360°$$

Hence $1 \text{ radian} = \dfrac{360°}{2\pi} \simeq 57.3°$.

Method for converting between degrees and radians

To convert degrees to radians, multiply by $\dfrac{2\pi}{360°} = \dfrac{\pi}{180°}$.

To convert radians to degrees, multiply by $\dfrac{360°}{2\pi} = \dfrac{180°}{\pi}$.

Some angles measured in radians can be written as simple fractions of π.

You must learn these.

Degrees	0°	15°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Where an angle is given without units, assume it is in radians.

Example

Convert $\dfrac{2\pi}{3}$ radians into degrees.

$\dfrac{\pi}{3} = 60°$ (see table) so $\dfrac{2\pi}{3} = 60° \times 2 = 120°$.

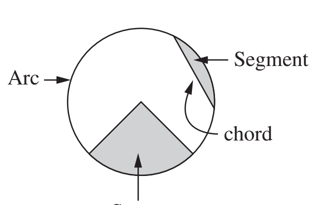
Example

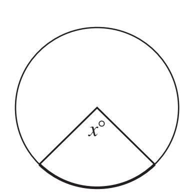
Convert 250° into radians.

This is not one of the commonly used angles (nor a multiple), so use the method for converting degrees to radians.

$250° \times \dfrac{\pi}{180°} = \dfrac{25\pi}{18} \simeq 4.36$

Circle sectors and segments





Considering the infinite rotational symmetry of the circle,

$$\frac{x°}{360°} = \frac{\text{arc length}}{2\pi r} = \frac{\text{sector area}}{\pi r^2}$$

That is, dividing the angle by 360°, the arc length by the circumference, and the sector area by the circle area gives the same fraction.

This is very useful when solving problems related to circles.

Changing the angles to radians gives formulae for the length of an arc and the area of a sector:

$$\frac{\theta}{2\pi} = \frac{\text{arc length}}{2\pi r}$$

$$\Rightarrow \text{arc length} = r\theta$$

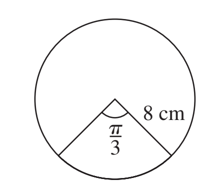
$$\frac{\theta}{2\pi} = \frac{\text{sector area}}{\pi r^2}$$

$$\Rightarrow \text{sector area} = \frac{1}{2}r^2\theta$$

These formulae only work if θ is in radians.

Example

What is the area of the sector shown below?



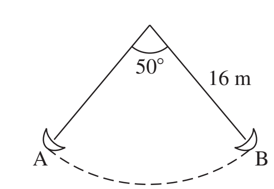
Sector area $= \dfrac{1}{2}r^2\theta$

$= \dfrac{1}{2} \times 8^2 \times \dfrac{\pi}{3}$

$= 33.5 \text{ cm}^2$

Example

The fairground ride shown below moves through an angle of 50° from point A to point B. What is the length of the arc AB?



Start by converting 50° into radians. $\theta = \dfrac{50°}{360°} \times 2\pi$

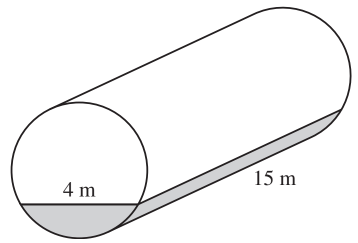
$= 0.872\ldots$

Hence arc length $= r\theta = 16 \times 0.872\ldots$

$= 13.96$ m

Example

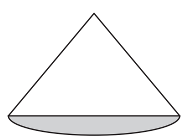
What is the volume of water lying in this pipe of radius 2.5 m?



In this example, we need to find the area of a segment. The method for doing this is:

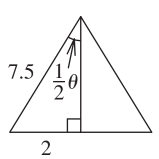
Area of segment = Area of sector − Area of triangle

It is important to remember this.



First find the angle at the centre in radians: $\frac{1}{2}\theta = \sin^{-1}\frac{2}{2.5}$

$= 0.927\ldots$



Area of triangle $= \frac{1}{2} \times 4 \times 1.5$

$= 3\ \text{m}^2$

Use Pythagoras to find the height of the triangle.

Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 2.5^2 \times 0.972\ldots \times 2$

$= 5.79\ldots\ \text{m}^2$

Area of segment = Area of sector − Area of triangle $= 5.79\ldots - 3$

$= 2.79\ldots\ \text{m}^2$

Volume $= 2.79\ldots \times 15 = 41.9\ \text{m}^3$

Exercise 1

1 Express each angle in degrees.

a $\frac{3\pi}{4}$ **b** $\frac{\pi}{9}$ **c** $\frac{2\pi}{5}$ **d** $\frac{5\pi}{6}$ **e** $\frac{7\pi}{12}$ **f** $\frac{\pi}{8}$

g $\frac{11\pi}{18}$ **h** 2 **i** 1.5 **j** 4 **k** 3.6 **l** 0.4

2 Express each angle in radians, giving your answer in terms of π.

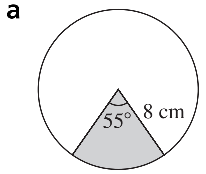
a 30° **b** 210° **c** 135° **d** 315°
e 240° **f** 70° **g** 72° **h** 54°

3 Express each angle in radians, giving your answer to 3 sf.

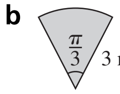
a 35° **b** 100° **c** 300°
d 80° **e** 132° **f** 278°

4 Find the area of each shaded sector.

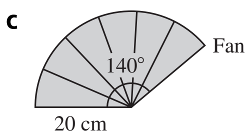
a



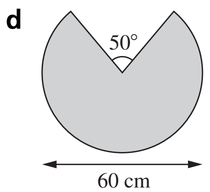
b



c

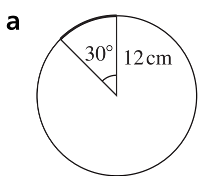


d

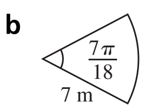


5 Find the length of each arc.

a



b



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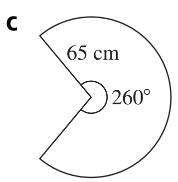
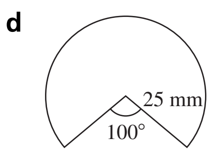
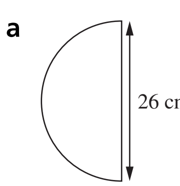


d

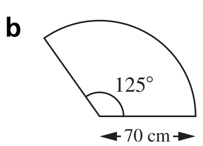


6 Find the perimeter of each shape.

a



b



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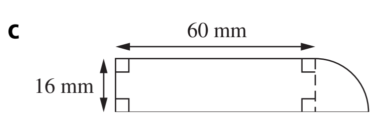
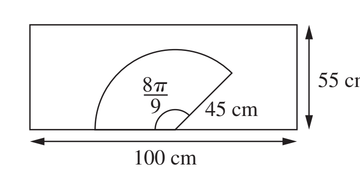


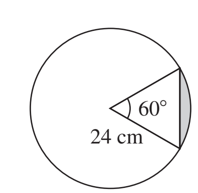
7 The diagram below shows a windscreen wiper cleaning a car windscreen.

a What is the length of the arc swept out?

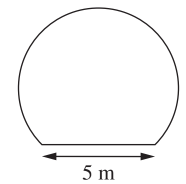
b What area of the windscreen is not cleared?



8 Find the area of the shaded segment.

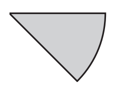


9 What is the area of this shape?

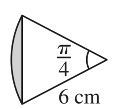


Diameter = 8 m

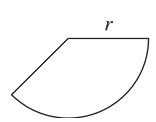
10 Radius = 32 cm
Area of sector = 1787 cm^2
What is the angle at the centre of the sector?



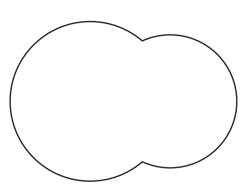
11 Find the perimeter of this segment.



12 A sector has an area of 942.5 cm^2 and an arc length of 62.8 cm. What is the radius of the circle?

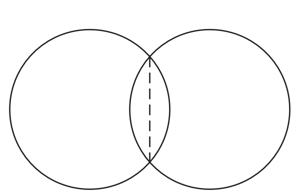


13 Two circles are used to form the logo for a company as shown below. One circle is of radius 12 cm. The other is of radius 9 cm. Their centres are 15 cm apart. What is the perimeter of the logo?



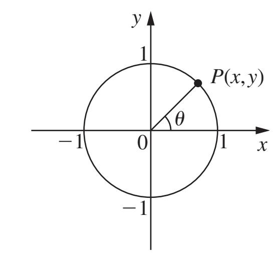
14 What is the ratio of the areas of the major sector in diagram A to the minor sector in diagram B?

15 Two circular table mats, each of radius 12 cm, are laid on a table with their centres 16 cm apart. Find
a the length of the common chord
b the area common to the two mats.

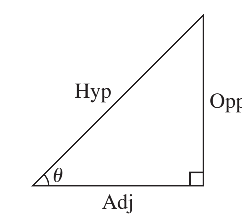


1.2 Trigonometric ratios

This unit circle can be used to define the trigonometric ratios.



You should already know that for a right-angled triangle



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

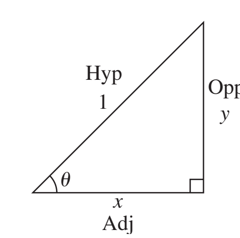
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

The x-coordinate is defined to be $\cos\theta$.

The y-coordinate is defined to be $\sin\theta$.

The results for a right-angled triangle follow from the definitions of the x- and y-coordinates in the unit circle.

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} \Rightarrow \tan\theta = \frac{y}{x}$$



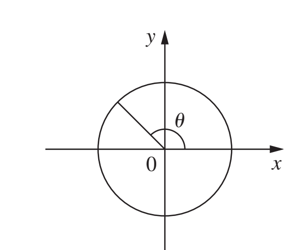
$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

This is the definition of $\tan\theta$ and is a useful identity.

More work will be done on trigonometric identities in Chapter 7.

Using the definition of $\sin\theta$ and $\cos\theta$ from the unit circle, we can see that these trigonometric ratios are defined not only for acute angles, but for any angle. For example, $\sin 120° = 0.866$ (3 sf).

As the x-coordinate is $\cos\theta$ and the y-coordinate is $\sin\theta$, for obtuse angles $\sin\theta$ is positive and $\cos\theta$ is negative.



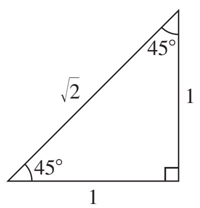
Exact values

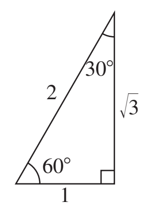
You need to learn sin, cos and tan of the angles given in the table overleaf for non-calculator examinations.

θ **(in radians)**	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
θ **(in degrees)**	$0°$	$30°$	$45°$	$60°$	$90°$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

The last row is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

These values can also be remembered using the triangles shown below.

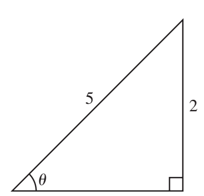




Finding an angle

When solving right-angled triangles, you found an acute angle.

Example

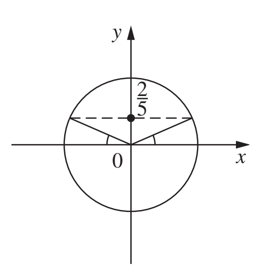


$\sin\theta = \frac{2}{5}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{2}{5}\right)$

$\Rightarrow \theta = 23.6°$

However, $\sin\theta = \frac{2}{5}$ has two possible solutions:



$\sin\theta = \frac{2}{5}$

$\Rightarrow \theta = 23.6°$ or $154.6°$

This is recognizing the symmetry of the circle.

Example

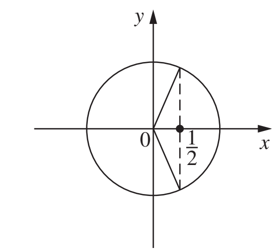
Solve $\cos\theta = \frac{1}{2}$ for $0 \le \theta < 2\pi$.

$\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$



Exercise 2

1 Find the value of each of these.

a $\sin 150°$ **b** $\sin 170°$ **c** $\cos 135°$ **d** $\cos 175°$

e $\sin\frac{2\pi}{3}$ **f** $\sin\frac{3\pi}{4}$ **g** $\cos\frac{5\pi}{6}$ **h** $\cos 2.4$(radians)

2 Without using a calculator, find the value of each of these.

a $\sin\frac{\pi}{6}$ **b** $\cos\frac{\pi}{3}$ **c** $\tan\frac{\pi}{4}$ **d** $\sin\frac{2\pi}{3}$

e $\cos\frac{5\pi}{3}$ **f** $\sin 135°$ **g** $\cos 315°$ **h** $\sin 180°$

i $\cos 180°$ **j** $\cos 270°$

3 Find the possible values of $x°$, given that $0° \le x° < 360°$.

a $\sin x° = \frac{1}{2}$ **b** $\cos x° = \frac{1}{3}$ **c** $\sin x° = \frac{2}{3}$

d $\cos x° = \frac{1}{6}$ **e** $\sin x° = \frac{3}{8}$ **f** $\cos x° = \frac{4}{7}$

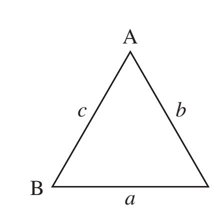
4 Find the possible values of θ, given that $0 \le \theta < 2\pi$.

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = \frac{1}{\sqrt{2}}$

d $\sin\theta = 2$ **e** $\cos\theta = \frac{\sqrt{3}}{2}$ **f** $\sin\theta = \frac{2}{7}$

g $\cos\theta = \frac{4}{11}$ **h** $\sin\theta = 0.7$

1.3 Solving triangles

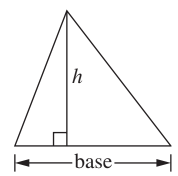


Vertices are given capital letters. The side opposite a vertex is labelled with the corresponding lower-case letter.

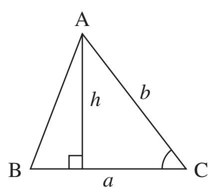
Area of a triangle

We know that the area of a triangle is given by the formula

$A = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$



To be able to use this formula, it is necessary to know the perpendicular height. This height can be found using trigonometry.



$\sin C = \frac{h}{b}$

$\Rightarrow h = b \sin C$

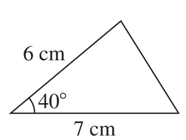
So the area of the triangle is given by

$$\text{Area} = \frac{1}{2} ab \sin C$$

This formula is equivalent to $\frac{1}{2} \times$ one side $\times$ another side $\times$ sine of angle between.

Example

Find the area of this triangle.

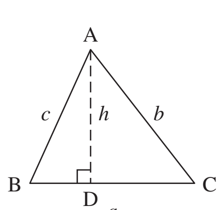


$\text{Area} = \frac{1}{2} \times 6 \times 7 \times \sin 40°$

$= 13.5 \text{ cm}^2$

Sine rule

Not all triangle problems can be solved using right-angled trigonometry. A formula called the sine rule is used in these problems.



$\sin B = \frac{AD}{AB}$ $\quad$ $\sin C = \frac{AD}{AC}$

$\Rightarrow AD = AB \times \sin B$ $\quad$ $\Rightarrow AD = AC \times \sin C$

$\Rightarrow AD = c \sin B$ $\quad$ $\Rightarrow AD = b \sin C$

$\Rightarrow c \sin B = b \sin C$

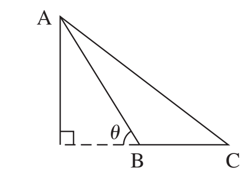
$\Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$

Drawing a line perpendicular to AC from B provides a similar result: $\frac{a}{\sin A} = \frac{c}{\sin C}$

Putting these results together gives the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Look at this obtuse-angled triangle:



If we consider the unit circle, it is clear that $\sin \theta = \sin(180° - \theta)$ and hence $\sin \theta = \sin B$. So the result is the same.

This is dealt with in more detail later in the chapter in relation to trigonometric graphs.

Use the sine rule in this form when finding a side:

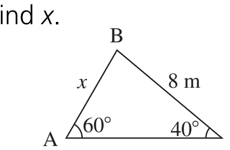
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use the sine rule in this form when finding an angle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example

Find x.



$\frac{a}{\sin A} = \frac{b}{\sin B}$

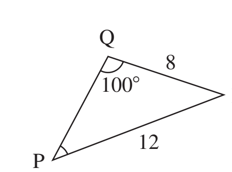
$\Rightarrow \frac{x}{\sin 40°} = \frac{8}{\sin 60°}$

$\Rightarrow x = \frac{8 \sin 40°}{\sin 60°}$

$\Rightarrow x = 5.94 \text{ m}$

Example

Find angle P.



$\frac{\sin P}{p} = \frac{\sin Q}{q}$

$\Rightarrow \frac{\sin P}{8} = \frac{\sin 100°}{12}$

$\Rightarrow \sin P = \frac{8 \sin 100°}{12}$

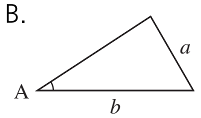
$\Rightarrow \sin P = 0.656\ldots$

$\Rightarrow P = 41.0°$

When the given angle is acute and it is opposite the shorter of two given sides, there are two possible triangles.

Example

In a triangle, angle A = 40°, $a = 9$ and $b = 13$. Find angle B.



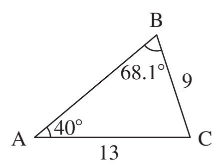
$$\frac{\sin 40°}{9} = \frac{\sin B}{13}$$

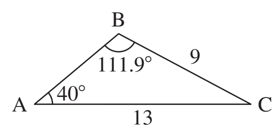
$$\Rightarrow \sin B = \frac{13 \sin 40°}{9}$$

$$\Rightarrow \sin B = 0.928\cdots$$

$$\Rightarrow B = 68.1° \text{ or } B = 180° - 68.1° = 111.9°$$

Hence it is possible to draw two different triangles with this information:





Cosine rule

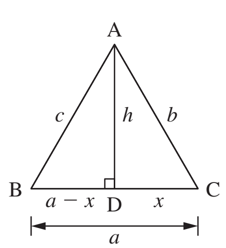
The sine rule is useful for solving triangle problems but it cannot be used in every situation. If you know two sides and the angle between them, and want to find the third side, the cosine rule is useful.

The cosine rule is

$$a^2 = b^2 + c^2 - 2bc \cos A$$

We can prove this using an acute-angled triangle:

We know that $h^2 = c^2 - (a - x)^2 = c^2 - a^2 + 2ax - x^2$ and $h^2 = b^2 - x^2$.



Hence $b^2 - x^2 = c^2 - a^2 + 2ax - x^2$

$$\Rightarrow c^2 = a^2 + b^2 - 2ax$$

Now $\cos C = \dfrac{x}{b}$

$$\Rightarrow x = b \cos C$$

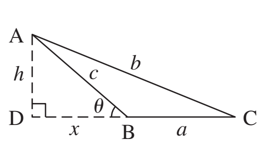
$$\Rightarrow c^2 = a^2 + b^2 - 2ab \cos C$$

Drawing the perpendicular from the other vertices provides different versions of the rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

The proof for an obtuse-angled triangle is similar:



In triangle ABD, $h^2 = c^2 - x^2$

In triangle ACD, $h^2 = b^2 - (a + x)^2$

$$= b^2 - a^2 - 2ax - x^2$$

$$\Rightarrow c^2 - x^2 = b^2 - a^2 - 2ax - x^2$$

$$\Rightarrow c^2 = b^2 - a^2 - 2ax$$

Now $\dfrac{x}{c} = \cos\theta$

$= \cos(180° - B)$

We will return to this later in the chapter.

$= -\cos B$

$\Rightarrow x = -c \cos B$

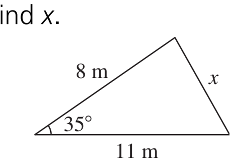
$$\Rightarrow c^2 = b^2 - a^2 + 2ac \cos B$$

$$\Rightarrow b^2 = a^2 + c^2 - 2ac \cos B$$

This situation is similar to the area of the triangle formula. The different forms do not need to be remembered: it is best thought of as two sides and the angle in between.

Example

Find x.



$$x^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos 35°$$

$$x^2 = 40.829\ldots$$

$$x = 6.39 \text{ m}$$

Pythagoras' theorem can be considered a special case of the cosine rule. This is the case where A = 90° ⇒ cos A = 0.

The cosine rule can be rearranged to find an angle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

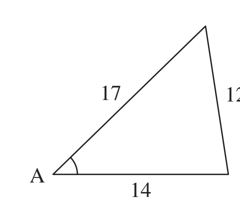
$$\Rightarrow 2bc \cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

This is only one form. It may be useful to re-label the vertices in the triangle.

Example

Find angle A.



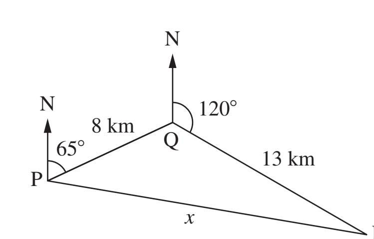
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \cos A = \frac{17^2 + 14^2 - 12^2}{2 \times 17 \times 14}$$

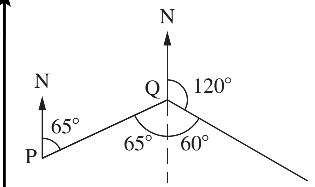
$$= 0.716\ldots$$

$$\Rightarrow A = 44.2°$$

Example



A ship sails on a bearing of 065° for 8 km, then changes direction at Q to a bearing of 120° for 13 km. Find the distance and bearing of R from P.



To find the distance x, angle Q is needed.

As the north lines are parallel, we can find angle Q.

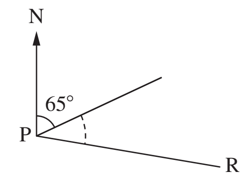
So $Q = 125°$

Using the cosine rule, $x^2 = 8^2 + 13^2 - 2 \times 8 \times 13 \times \cos 125°$

$x^2 = 352.3\cdots$

$\Rightarrow x = 18.8$ km

We can now find the bearing of R from P.



Using the sine rule, $\dfrac{\sin P}{13} = \dfrac{\sin 125°}{18.8}$

$\Rightarrow \sin P = \dfrac{13 \sin 125°}{18.8} = 0.566\cdots$

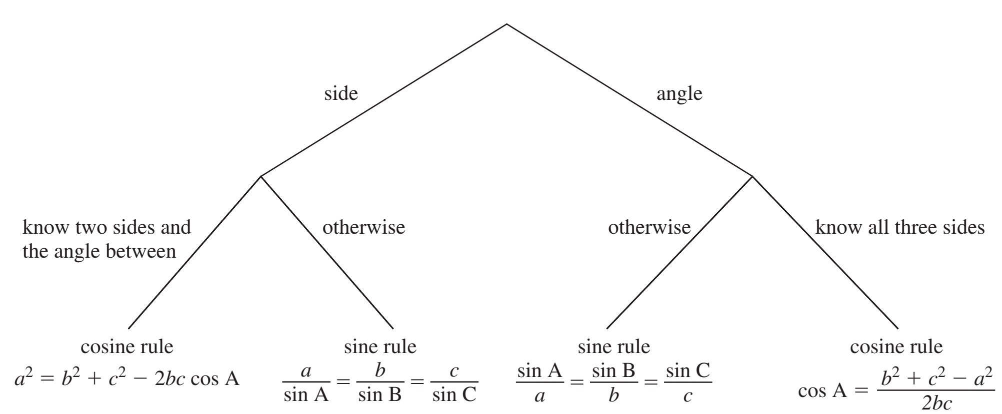
$\Rightarrow P = 34.5°$

Bearing of R from P is $65 + 34.5 = 099.5°$.

Decision making about triangle problems

It is worth remembering that Pythagoras' theorem and right-angled trigonometry can be applied to right-angled triangles, and they should not need the use of the sine rule or the cosine rule.

For non-right angled triangles, use this decision tree.

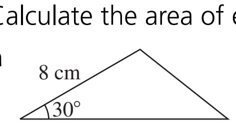


Once two angles in a triangle are known, the third angle can be found by subtracting the other two angles from 180°.

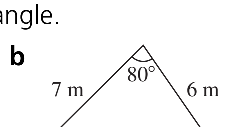
Exercise 3

1 Calculate the area of each triangle.

a



b



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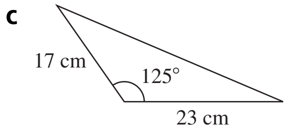
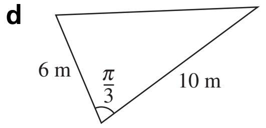
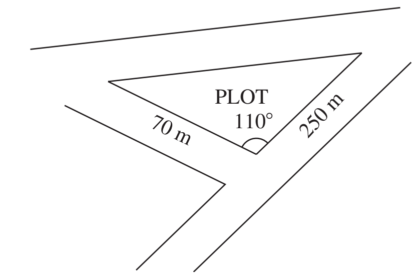


d



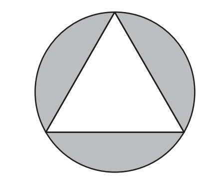
2 Three roads intersect as shown, with a triangular building plot between them. Calculate the area of the building plot.



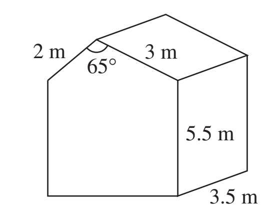
3 A design is created by an equilateral triangle of side 14 cm at the centre of a circle.

a Find the area of the triangle.

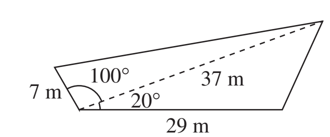
b Hence find the area of the segments.



4 An extension to a house is built as shown.
What is the volume of the extension?

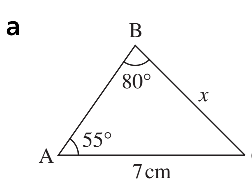


5 Find the area of this campsite.



6 Use the sine rule to find the marked side.

a



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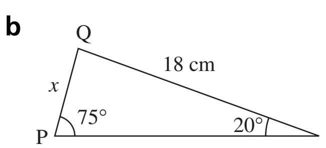

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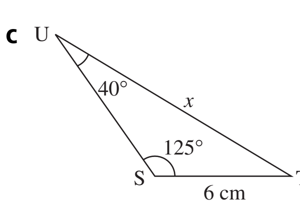

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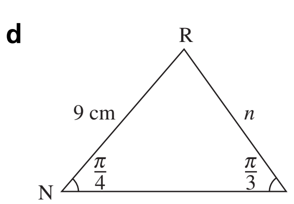
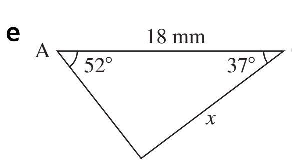
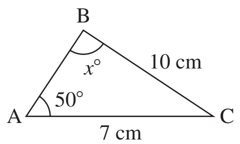


e



7 Use the sine rule to find the marked angle.

a



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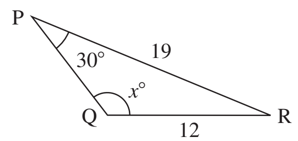
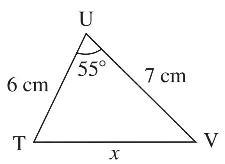


8 Triangle LMN has sides LM = 32 m and MN = 35 m with LNM = 40°
Find the possible values for ∠MLN.

9 Triangle ABC has sides AB = 11 km and BC = 6 km and ∠BAC = 20°.
Calculate ∠BCA.

10 Use the cosine rule to find the marked side.

a



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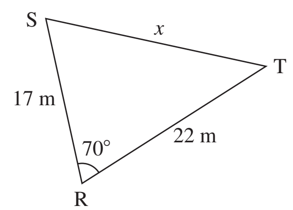

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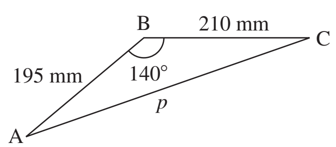

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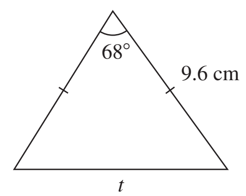
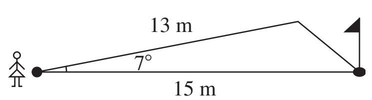
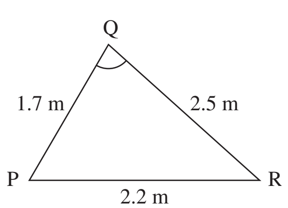


11 A golfer is standing 15 m from the hole. She putts 7° off-line and the ball travels 13 m. How far is her ball from the hole?



12 Use the cosine rule to find the marked angle.

a



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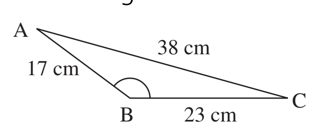
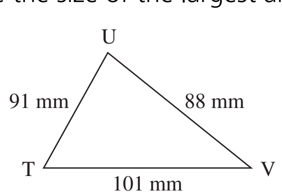
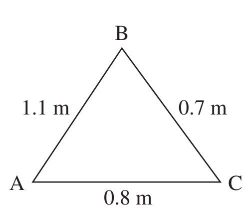


13 Calculate the size of the largest angle in triangle TUV.

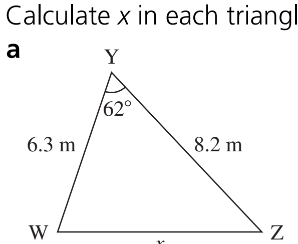


14 Find the size of all the angles in triangle ABC.

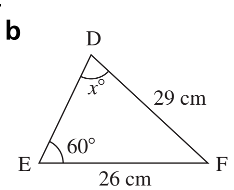


15 Calculate x in each triangle.

a



b



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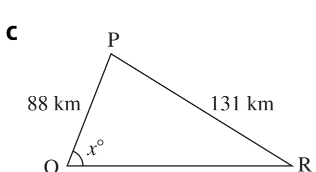

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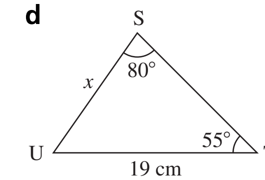

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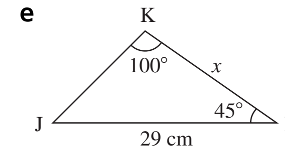
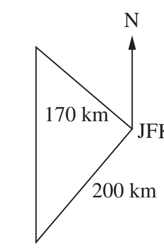
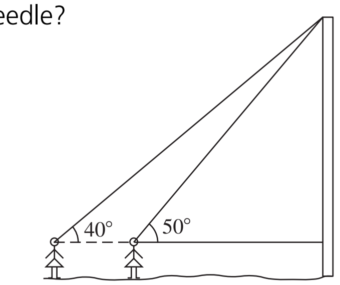


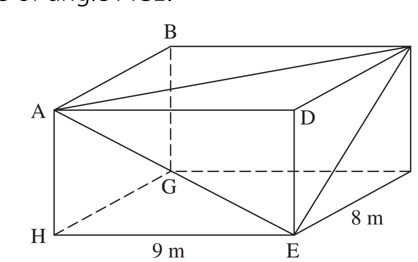
16 A plane flies from New York JFK airport on a bearing of 205° for 200 km. Another plane also leaves from JFK and flies for 170 km on a bearing of 320°. What distance are the two planes now apart?



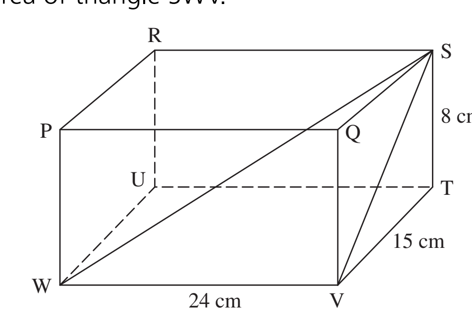
17 Twins Anna and Tanya, who are both 1.75 m tall, both look at the top of Cleopatra's Needle in Central Park, New York. If they are standing 7 m apart, how tall is the Needle?



18 Find the size of angle ACE.



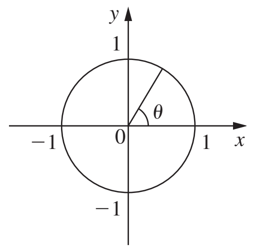
19 Find the area of triangle SWV.



1.4 Trigonometric functions and graphs

$\sin\theta$ is defined as the y-coordinate of points on the unit circle.

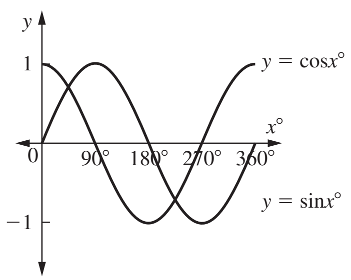
θ	0°	30°	45°	60°	90°	180°	270°	360°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0



$\cos\theta$ is defined as the x-coordinate of points on the unit circle.

θ	0°	30°	45°	60°	90°	180°	270°	360°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1

These functions are plotted below.

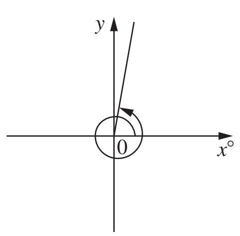


Both graphs only have y-values of $-1 \le y \le 1$.

Periodicity

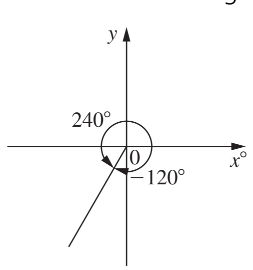
When considering angles in the circle, it is clear that any angle has an equivalent angle in the domain $0 \le x° < 360°$.

For example, an angle of 440° is equivalent to an angle of 80°.



$440° = 360° + 80°$

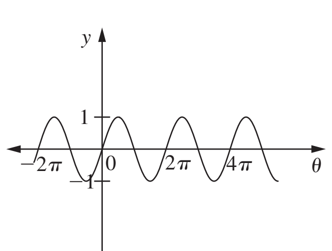
This is also true for negative angles.



$-120° = 240°$

This means that the sine and cosine graphs are infinite but repeat every 360° or 2π.

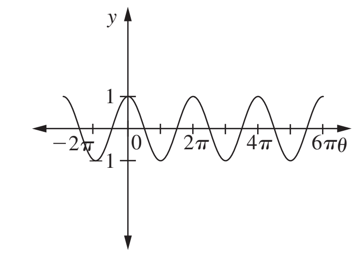
$y = \sin\theta$



These graphs can be drawn using degrees or radians.

1 Trigonometry 1

$y = \cos\theta$



Repeating at regular intervals is known as **periodicity**. The period is the interval between repetitions.

For $y = \sin\theta$ and $y = \cos\theta$, the period is 360° or 2π.

Graph of tan x°

We have defined $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. This allows us to draw its graph.

θ	0°	30°	45°	60°	90°	180°	270°	360°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0	undefined	0

The vertical asymptote is a line: there are other types of asymptote that we will meet later.

There is a problem when $x° = 90°, 270° \ldots$ because there is a zero on the denominator. This is undefined (or infinity). Graphically, this creates a **vertical asymptote**. This is created by an x-value where the function is not defined. The definition of an asymptote is that it is a line associated with a curve such that as a point moves along a branch of the curve, the distance between the line and the curve approaches zero. By examining either side of the vertical asymptote, we can obtain the behaviour of the function around the asymptote.

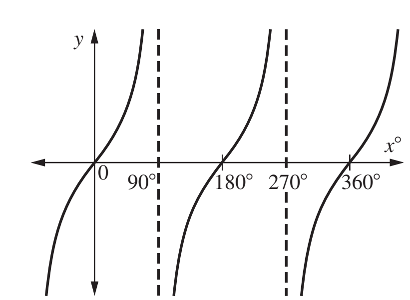
As $x° \to 90°$ (x° approaches 90°) tan x° increases and approaches ∞ (infinity):

$\tan 85° = 11.4$, $\tan 89° = 57.3$, $\tan 89.9° = 573$ etc.

On the other side of the asymptote, tan x° decreases and approaches $-\infty$:

$\tan 95° = -11.4$, $\tan 91° = -57.3$, $\tan 90.1° = -573$ etc.

The graph of $y = \tan x°$ is shown below.



It is clear that this graph is also periodic, and the period is 180°.

Reciprocal trigonometric functions

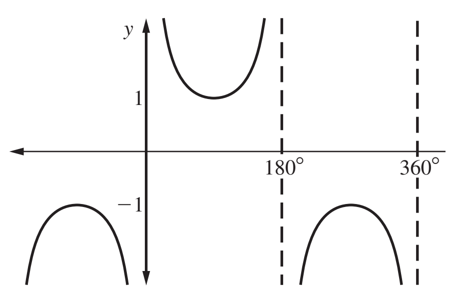
There are three more trigonometrical functions, defined as the reciprocal trigonometric functions – secant, cosecant and cotangent. Secant is the reciprocal function to cosine, cosecant is the reciprocal function to sine, and cotangent is the reciprocal function to tangent. These are abbreviated as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}\ (\text{or cosec}\ \theta) \quad \cot\theta = \frac{1}{\tan\theta}$$

In order to obtain the graph of $f(x) = \csc\theta$, consider the table below.

$\boldsymbol{\theta}$	0°	30°	45°	60°	90°	180°	270°	360°
sin $\boldsymbol{\theta}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
csc $\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{sin}\ \boldsymbol{\theta}}$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞

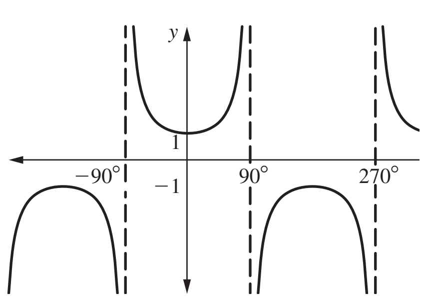
The roots (zeros) of the original function become vertical asymptotes in the reciprocal function.



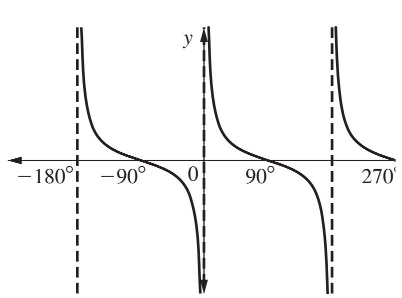
$y = \csc\theta$

This function is also periodic with a period of 360°.

Similarly we can obtain the graphs of $y = \sec\theta$ and $y = \cot\theta$:



$y = \sec\theta$



$y = \cot\theta$

The general method for plotting reciprocal graphs will be addressed in Chapter 8.

Composite graphs

Using your graphing calculator, draw the following graphs to observe the effects of the transformations.

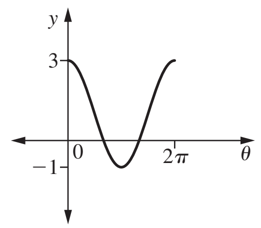
1. $y = 2\sin x°$
   $y = 3\cos x°$
   $y = 5\sin x°$
   $y = \frac{1}{2}\sin x°$
2. $y = \sin 2x°$
   $y = \cos 3x°$
   $y = \sin 5x°$
   $y = \cos\frac{1}{2}x°$
3. $y = -\sin x°$
   $y = -\cos x°$
   $y = -\tan x°$
4. $y = \sin(-x°)$
   $y = \cos(-x°)$
   $y = \tan(-x°)$
5. $y = \sin x° + 2$
   $y = \cos x° + 2$
   $y = \sin x° - 1$
   $y = \cos x° - 1$
6. $y = \sin(x - 30)°$
   $y = \cos(x - 30)°$
   $y = \sin\left(\theta + \frac{\pi}{3}\right)$
   $y = \cos\left(\theta + \frac{\pi}{3}\right)$

The table summarizes the effects.

$\boldsymbol{y} =$	**Effect**	**Notes**
$A\sin x$, $A\cos x$, $A\tan x$	Vertical stretch	
$\sin Bx$, $\cos Bx$, $\tan Bx$	Horizontal stretch/ compression	This is the only transformation that affects the period of the graph
$-\sin x$, $-\cos x$, $-\tan x$	Reflection in x-axis	
$\sin(-x)$, $\cos(-x)$, $\tan(-x)$	Reflection in y-axis	
$\sin x + C$, $\cos x + C$, $\tan x + C$	Vertical shift	
$\sin(x + D)$, $\cos(x + D)$, $\tan(x + D)$	Horizontal shift	Positive D left, negative D right

Example

Draw the graph of $y = 2\cos\theta + 1$ for $0 \le \theta < 2\pi$.

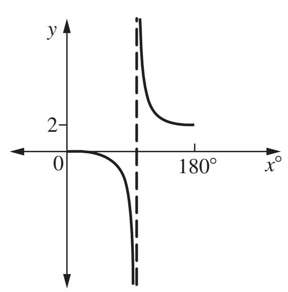


This is a vertical stretch $\times 2$ and a vertical shift $+1$.

The domain tells you how much of the graph to draw and whether to work in degrees or radians.

Example

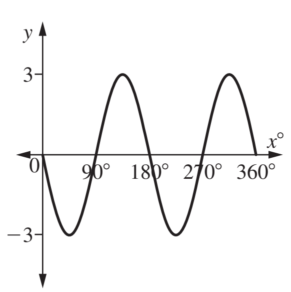
Draw the graph of $y = \csc(x - 90)° + 1$ for $0° \le x° < 180°$.



This is a horizontal shift of 90° to the right and a vertical shift $+1$.

Example

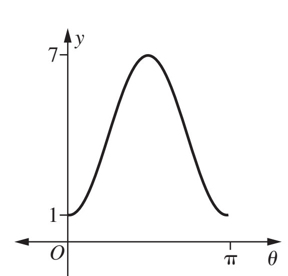
Draw the graph of $y = -3\sin 2x°$ for $0° \le x° < 360°$.



In this case, there are two full waves in 360°. There is a reflection in the x-axis and a vertical stretch $\times 3$.

Example

What is the equation of this graph?



We assume that, because of the shape, it is either a sine or cosine graph. Since it begins at a minimum point, we will make the assumption that it is a cosine graph. (We could use sine but this would involve a horizontal shift, making the question more complicated.) $y = \cos\theta$

Since it is "upside down" there is a reflection in the x axis $\Rightarrow y = -\cos\theta$

There is a period of π and so there are two full waves in $2\pi \Rightarrow y = -\cos 2\theta$

There is a difference of 6 between the max and min values. There would normally be a difference of 2 and hence there is a $\times 3$ vertical stretch

$\Rightarrow y = -3\cos 2\theta$

The min and max values are 1, 7 so there is a shift up of 4

$\Rightarrow y = -3\cos 2\theta + 4$

So the equation of this graph is $y = -3\cos 2\theta + 4$.

Exercise 4

1 What is the period of each function?

a $y = \sin 2x°$ **b** $y = \cos 3x°$ **c** $y = \cos 4\theta + 1$
d $y = \tan 2x°$ **e** $y = \sec x°$ **f** $y = 2\cos 3x° - 3$
g $y = 5\csc 2x° + 3$ **h** $y = 7 - 3\sin 4\theta$ **i** $y = 9\sin 10x°$
j $y = 8\tan 60x°$

2 Draw the graphs of these functions for $0° \le x° < 360°$.

a $y = \sin 3x°$ **b** $y = -\cos x°$ **c** $y = \sin(-x°)$
d $y = 4\csc x$ **e** $y = \tan(x - 30)°$ **f** $y = \sec x° + 2$

3 Draw the graphs of these functions for $0 \le \theta < 2\pi$.

a $y = -\sin 2\theta$ **b** $y = \cot\left(\theta + \frac{\pi}{3}\right)$ **c** $y = 4\cos\theta - 2$
d $y = \csc(-\theta)$ **e** $y = 3 - 5\sin\theta$

4 Draw the graphs of these functions for $0° \le x° < 180°$.

a $y = \cos 3x°$ **b** $y = 2\sin 4x°$ **c** $y = 3\sec 2x°$
d $y = 6\sin 10x°$ **e** $y = 2\tan(x + 30)°$ **f** $y = 3\cos 2x° - 1$

5 Draw the graphs of these functions for $0 \le \theta < \pi$.

a $y = 6\cos\theta + 2$ **b** $y = 3\sin 4\theta - 5$ **c** $y = 7 - 4\cos\theta$
d $y = \cot 3\theta$ **e** $y = \tan\left(2\theta - \frac{\pi}{3}\right)$

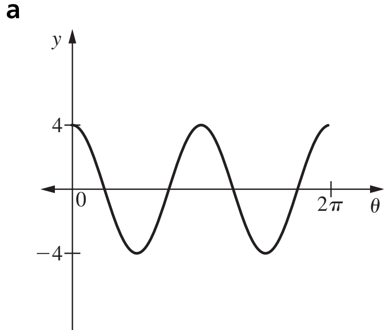
6 Draw the graph of $y = 4\sin 2x° - 3$ for $0° \le x° < 720°$.

7 Draw the graph of $y = 6\cos 30x°$ for $0° \le x° < 12°$.

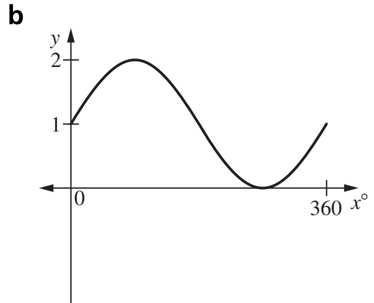
8 Draw the graph of $y = 8 - 3\sin 4\theta$ for $0 \le \theta < \frac{\pi}{2}$.

9 Find the equation of each graph.

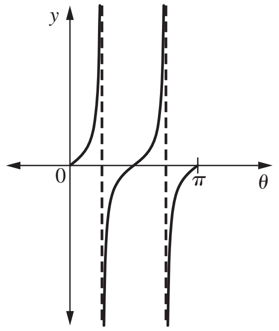
a



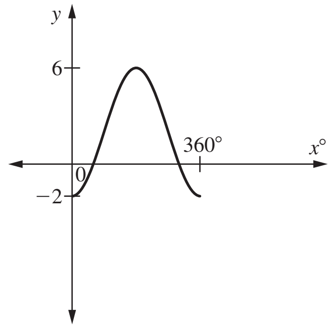
b



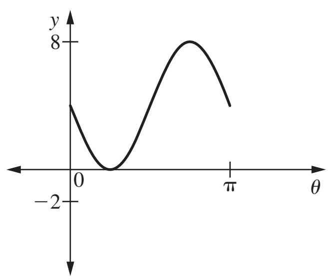
c



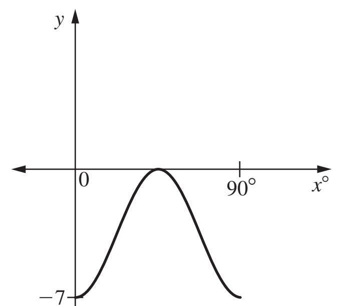
d



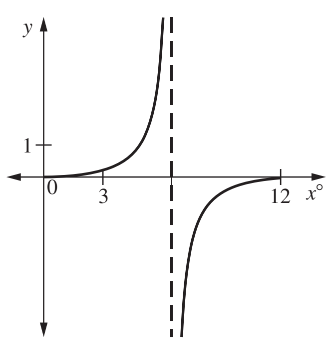
e



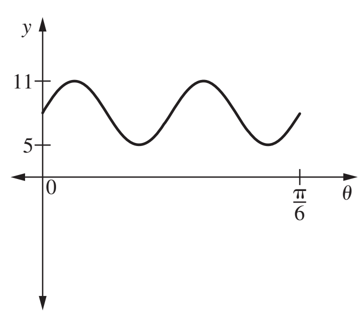
f



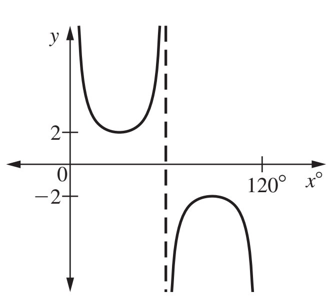
g



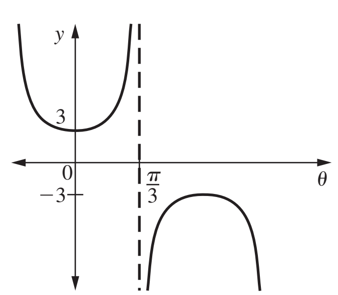
h



i



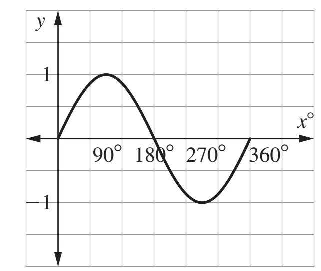
j



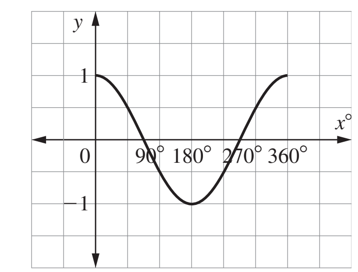
1.5 Related angles

To be able to solve trigonometric equations algebraically we need to consider properties of the trigonometric graphs. Each graph takes a specific y-value for an infinite number of x-values. Within this curriculum, we consider this only within a finite domain. Consider the graphs below, which have a domain $0° \le x° < 360°$.

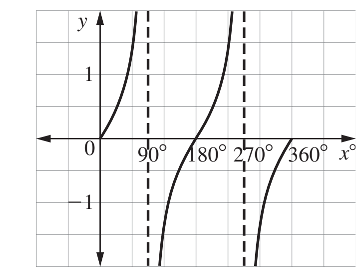
$y = \sin x°$



$y = \cos x°$



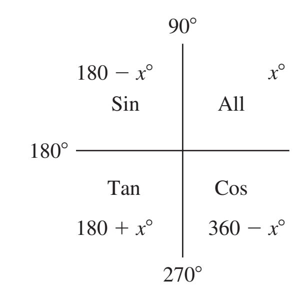
$y = \tan x°$



These graphs can be split into four quadrants, each of 90°. We can see that in the first quadrant all three graphs are above the x-axis (positive).

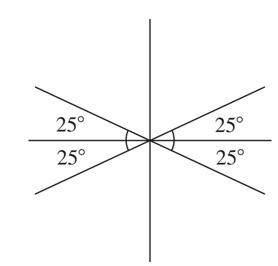
In each of the other three quadrants, only one of the functions is positive.

This is summarised in the following diagram.



The diagram shows two important features. First, it shows where each function is positive. Second, for every acute angle, there is a related angle in each of the other three quadrants. These related angles give the same numerical value for each trigonometric function, ignoring the sign. This diagram is sometimes known as the **bow-tie diagram**.

By taking an example of 25°, we can see all of the information that the bow-tie diagram provides:



Related angles

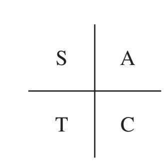
25°

$180 - 25 = 155°$

$180 + 25 = 205°$

$360 - 25 = 335°$

Using



we can say that

$\sin 155° = \sin 25°$

$\sin 205° = -\sin 25°$

$\sin 335° = -\sin 25°$

$\cos 155° = -\cos 25°$

$\cos 205° = -\cos 25°$

$\cos 335° = \cos 25°$

$\tan 155° = -\tan 25°$

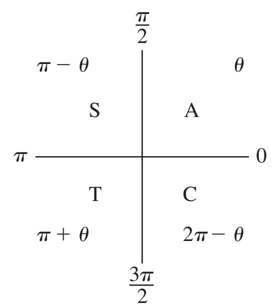
$\tan 205° = \tan 25°$

$\tan 335° = -\tan 25°$

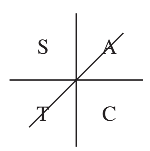
Example

Find the exact value of $\cos\frac{4\pi}{3}$, $\sin\frac{7\pi}{6}$ and $\tan\frac{7\pi}{4}$.

This is the bow-tie diagram in radians:



For $\cos\frac{4\pi}{3}$, we need to find the related acute angle.



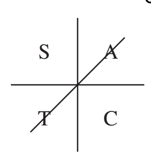
$$\pi + \theta = \frac{4\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Since $\frac{4\pi}{3}$ is in the third quadrant, $\cos\frac{4\pi}{3}$ is negative.

So $\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$

Considering $\sin\frac{7\pi}{6}$:



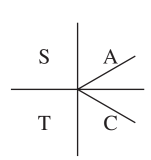
$$\pi + \theta = \frac{7\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Since $\frac{7\pi}{6}$ is in the third quadrant, $\sin\frac{7\pi}{6}$ is negative.

So $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

Considering $\tan\frac{7\pi}{4}$:



$$2\pi - \theta = \frac{7\pi}{4}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Since $\frac{7\pi}{4}$ is in the fourth quadrant, $\tan\frac{7\pi}{4}$ is negative.

So $\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4} = -1$

Exercise 5

1 Find the exact value of each of these.

a $\cos 120°$ **b** $\tan 135°$ **c** $\sin 150°$ **d** $\cos 300°$
e $\tan 225°$ **f** $\cos 210°$ **g** $\tan 300°$ **h** $\sin 240°$
i $\cos 330°$ **j** $\cos 150°$

2 Find the exact value of each of these.

a $\tan\frac{7\pi}{6}$ **b** $\sin\frac{3\pi}{4}$ **c** $\cos\frac{11\pi}{6}$ **d** $\tan\frac{5\pi}{3}$
e $\sin\frac{5\pi}{4}$ **f** $\tan\frac{5\pi}{6}$ **g** $\cos\frac{3\pi}{2}$ **h** $\sin\frac{5\pi}{3}$
i $2\sin\frac{5\pi}{6}$ **j** $8\cos\frac{11\pi}{6}$

3 Express the following angles, using the bow-tie diagram, in terms of the related acute angle.

a $\sin 137°$ **b** $\cos 310°$ **c** $\tan 200°$
d $\sin 230°$ **e** $\cos 157°$ **f** $\tan 146°$
g $\cos 195°$ **h** $\sin 340°$ **i** $\tan 314°$

4 State two possible values for $x°$ given that $0° \le x° < 360°$.

a $\sin x° = \frac{1}{2}$ **b** $\cos x° = \frac{\sqrt{3}}{2}$
c $\tan x° = \sqrt{3}$ **d** $\tan x° = -1$

5 State two possible values for θ given that $0 \le \theta < 2\pi$.

a $\sin\theta = \frac{\sqrt{3}}{2}$ **b** $\tan\theta = \frac{1}{\sqrt{3}}$
c $\cos\theta = \frac{1}{2}$ **d** $\cos\theta = -\frac{\sqrt{3}}{2}$

1.6 Trigonometric equations

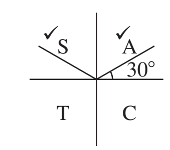
We can use related angles to help solve trigonometric equations, especially without a calculator.

Example

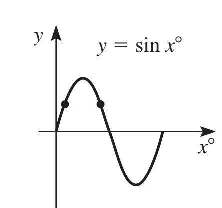
Solve $2\sin x° + 3 = 4$ for $0° \le x° < 360°$.

It is very important to take account of the domain.

$$2\sin x° + 3 = 4$$
$$\Rightarrow 2\sin x° = 1$$
$$\Rightarrow \sin x° = \frac{1}{2}$$
$$\Rightarrow x° = 30°, (180 - 30)°$$
$$\Rightarrow x° = 30°, 150°$$



Thinking of the graph of $\sin x°$, it is clear these are the only two answers:

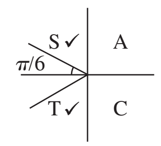


Example

Solve $2\cos\theta + \sqrt{3} = 0$ for $0 \le \theta < 2\pi$.

$2\cos\theta + \sqrt{3} = 0$

$\Rightarrow \cos\theta = -\dfrac{\sqrt{3}}{2}$



We know that $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and cos is negative in the second and third quadrants.

$\Rightarrow \theta = \left(\pi - \dfrac{\pi}{6}\right), \left(\pi + \dfrac{\pi}{6}\right)$

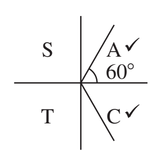
$\Rightarrow \theta = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

Example

Solve $2\cos(3x - 15)° = 1$ for $0° \le x° < 360°$.

$2\cos(3x - 15)° = 1$

$\Rightarrow \cos(3x - 15)° = \dfrac{1}{2}$



$\Rightarrow (3x - 15)° = 60°$ or $300°$

$\Rightarrow 3x° = 75°$ or $315°$

$\Rightarrow x° = 25°$ or $105°$

We know that $3x$ means three full waves in 360° and so the period is 120°.

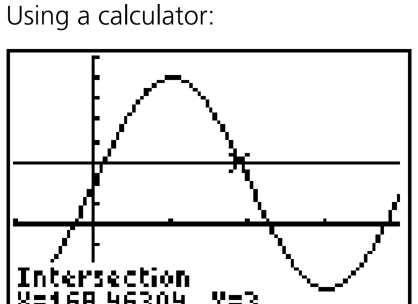
The other solutions can be found by adding on the period to these initial values:
$x° = 25°, 105°, 145°, 225°, 265°, 345°$

A graphical method can also be used to solve trigonometric equations, using a calculator.

Example

Solve $5\sin x° + 2 = 3$ for $0° \le x° < 360°$.

Using a calculator:

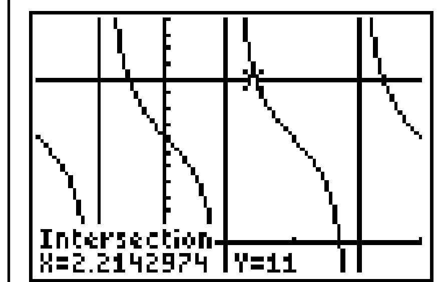


$x° = 11.5°, 168.5°$

1 Trigonometry 1

Example

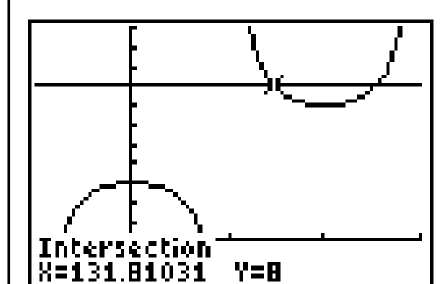
Solve $7 - 3\tan\theta = 11$ for $0 \le \theta < 2\pi$.



$\theta = 2.21, 5.36$

Example

Solve $5 - 2\sec x° = 8$ for $0° \le x° < 180°$.



Noting the domain, $x° = 131.8°$

The algebraic method can be used in conjunction with a calculator to solve any equation.

Example

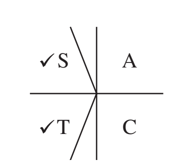
Solve $3\cos 3\theta + 5 = 4$ for $0 \le \theta < 2\pi$.

$3\cos 3\theta + 5 = 4$

$\Rightarrow 3\cos 3\theta = -1$

$\Rightarrow \cos 3\theta = -\dfrac{1}{3}$

Use a calculator to find $\cos^{-1}\left(\dfrac{1}{3}\right) = 1.23$



Cos is negative in the second and third quadrants.

$\Rightarrow 3\theta = \pi - 1.23, \pi + 1.23$

$\Rightarrow 3\theta = 1.91, 4.37$

$\Rightarrow \theta = 0.637, 1.46$

Here the period is $\frac{2\pi}{3}$ and hence we can find all six solutions:

$\theta = 0.637, 1.46, 2.73, 3.55, 4.83, 5.65$

Example

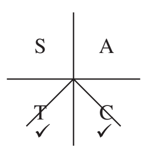
Solve $2 \sin 2\theta + 3 = 2$ for $-\pi \le \theta < \pi$.

Here we notice that the domain includes negative angles. It is solved in the same way.

$2 \sin 2\theta + 3 = 2$

$\Rightarrow 2 \sin 2\theta = -1$

$\Rightarrow \sin 2\theta = -\frac{1}{2}$



sin is negative in the third and fourth quadrants:

We know that $\sin \frac{\pi}{6} = \frac{1}{2}$

so the related angles are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Hence $2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\Rightarrow \theta = \frac{7\pi}{12}, \frac{11\pi}{12}$

Now we just need to ensure that we have all of the solutions within the domain by using the period. These two solutions are both within the domain. The other two solutions required can be found by subtracting a period:

$\theta = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

Exercise 6

1 Solve these for $0° \le x° < 360°$.

a $\tan x° = \sqrt{3}$ **b** $\cos x° = \frac{1}{2}$ **c** $\sin x° = \frac{\sqrt{3}}{2}$

d $2 \sin x° + 1 = 0$ **e** $2 \cos x° = -\sqrt{3}$ **f** $\cos x° + 1 = 0$

g $4 \sin x° - 3 = 1$ **h** $\csc x° = 2$ **i** $6 \cot x° - 1 = 5$

2 Solve these for $0 \le \theta < 2\pi$.

a $\cos \theta = \frac{\sqrt{3}}{2}$ **b** $\sin \theta = -\frac{1}{2}$ **c** $\tan \theta = -\frac{1}{\sqrt{3}}$

d $3 \tan \theta + 2 = 5$ **e** $4 - 2 \sin \theta = 3$ **f** $3 \tan \theta = \sqrt{3}$

g $\sin\left(\theta - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ **h** $\sqrt{3} \sec \theta = 2$

3 Solve these for $0° \le x° < 360°$.

a $\sin 2x° = \frac{1}{2}$ **b** $2 \cos 3x° = \sqrt{3}$ **c** $6 \tan 4x° = 6$

d $2 \cos 2x° = -1$ **e** $4 \sin(3x - 15)° = 2\sqrt{3}$ **f** $\sec 3x° = -2$

4 Solve these for $0 \le \theta < 2\pi$.

a $\cos 4\theta = \frac{1}{2}$ **b** $\tan\left(2\theta - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

c $4 - 2 \sin 5\theta = 3$ **d** $6 \cos 2\theta = -\sqrt{27}$

5 Solve $\sqrt{3} \tan 2x° - 1 = 0$ for $0° \le x° < 180°$.

6 Solve $2 \sin 4\theta + 1 = 0$ for $0 \le \theta < \pi$.

7 Solve $6 \sin 30x° - 3 = 0$ for $0° \le x° < 24°$.

8 Solve $2 \tan x° = \sqrt{12}$ for $-180° \le x° < 180°$.

9 Solve $6 \cos 3\theta + 2 = -1$ for $-\pi \le \theta < \pi$.

10 Solve these for $0° \le x° < 360°$.

a $3 \sin x° = 1$ **b** $4 \cos x° = 3$ **c** $5 \tan x° - 1 = 7$

d $6 \cos x° - 5 = -1$ **e** $4 \sin x° - 3 = 0$ **f** $8 \cos(x + 20)° = 5$

g $3 - 4 \cos x° = 2$ **h** $\sqrt{2} \sin x° - 3 = -2$ **i** $9 \sin(x - 15)° = -5$

j $7 - 5 \sin x° = 4$ **k** $6 \sin 2x° - 5 = -1$ **l** $8 \cos 3x° + 5 = 7$

m $7 + 11 \tan 5x° = -9$ **n** $\sec x° = 3$ **o** $\csc x° - 2 = 5$

p $4 \sec x° + 3 = 9$ **q** $6 \cot x° - 1 = 8$ **r** $9 \sec 4x° + 3 = 21$

s $4 - 3 \sin 30x° = 2$

11 Solve these for $0 \le \theta < 2\pi$.

a $4 \sin \theta = 1$ **b** $9 \cos \theta = -4$ **c** $8 \tan \theta - 2 = 17$

d $\sqrt{5} \cos \theta - 4 = -3$ **e** $7 \cos\left(\theta - \frac{\pi}{3}\right) = 4$ **f** $6 - 5 \sin \theta = 7$

g $9 + 5 \tan \theta = 23$ **h** $3 \cos 3\theta - 1 = 0$ **i** $6 \sin 2\theta = -1$

j $7 - 2 \tan 4\theta = 13$ **k** $9 - 4 \sin 3\theta = 6$ **l** $8 \sec \theta = 19$

m $1 - 3 \csc \theta = 11$ **n** $2 + \cot \theta = 9$ **o** $6 \csc 4\theta - 3 = 11$

12 Solve $8 - 3 \cos x° = 7$ for $0° \le x° < 720°$.

13 Solve $5 + 2 \sin\left(3\theta - \frac{\pi}{4}\right) = 6$ for $-\pi \le \theta < \pi$.

14 The height of a basket on a Ferris wheel is modelled by

$$H(t) = 21 - 18 \sin\left(\frac{2\pi}{3}t\right)$$

where H is the height above the ground in metres and t is the time in minutes.

a How long does it take to make one complete revolution?

b Sketch the graph of the height of the basket during one revolution.

c When is the basket at its (i) maximum height (ii) minimum height?

15 The population of tropical fish in a lake can be estimated using

$$P(t) = 6000 + 1500 \cos 15t$$

where t is the time in years. Estimate the population

a initially **b** after 3 years.

c Find the minimum population estimate and when this occurs.

1.7 Inverse trigonometric functions

In order to solve trigonometric equations, we employed the inverse function.

For example, $\sin x° = \frac{1}{2}$

$$\Rightarrow x° = \arcsin\left(\frac{1}{2}\right)$$

arcsin is the inverse sine function (also denoted $\sin^{-1}$).

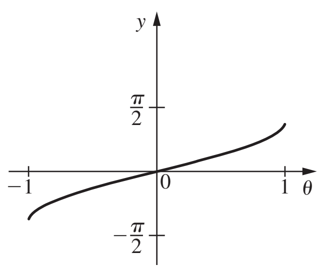
More work is done on inverse functions in Chapter 3.

An inverse function is one which has the opposite effect to the function itself.

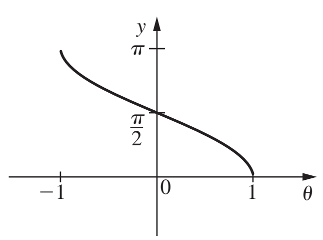
For an inverse function, the range becomes the domain and the domain becomes the range.

Hence for the inverse of the sine and cosine functions, the domain is $[-1, 1]$.

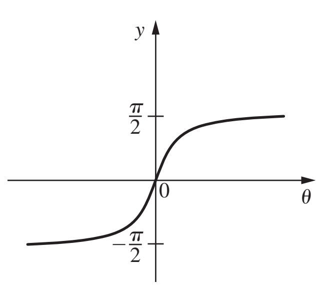
The graphs of the inverse trigonometric functions are:



$y = \arcsin\theta$



$y = \arccos\theta$



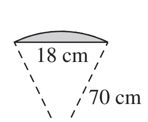
$y = \arctan\theta$

Review exercise

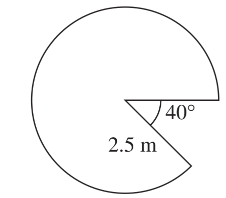
1 Express in degrees: **a** $\frac{\pi}{6}$ **b** $\frac{5\pi}{12}$

2 Express in radians: **a** 120° **b** 195°

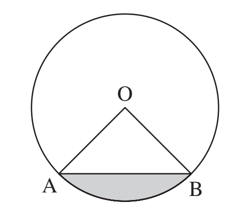
3 Find the area of this segment.



4 Find the length of this arc



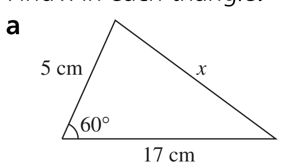
5 The diagram below shows a circle centre O and radius OA = 5 cm.
The angle AOB = 135°.
Find the shaded area.



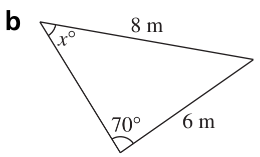
[IB Nov 04 P1 Q9]

6 Find x in each triangle.

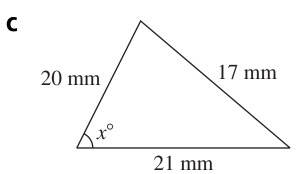
a



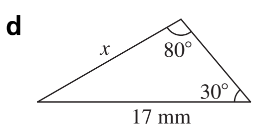
b



c



d



7 In the triangle ABC, the side AB has length 5 and the angle BAC = 28°. For what range of values of the length of BC will two distinct triangles ABC be possible?

8 Find the exact value of each of these.

a $\cos\frac{2\pi}{3}$ **b** $\sin\frac{3\pi}{4}$ **c** $\tan\frac{5\pi}{6}$ **d** $\sin\frac{7\pi}{6}$

e $\cos\frac{7\pi}{4}$ **f** sin 300° **g** tan 240° **h** cos 135°

i tan 330° **j** sec 60° **k** csc 240°

9 Sketch each of these graphs.

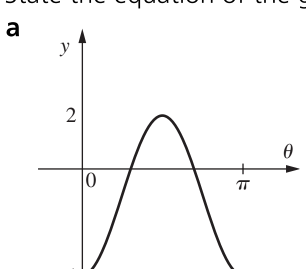
a $y = 6\cos 2\theta - 1$ **b** $y = 4\sin(x - 30)°$

c $y = 8 - 3\cos 4\theta$ **d** $y = 5\sec\theta$

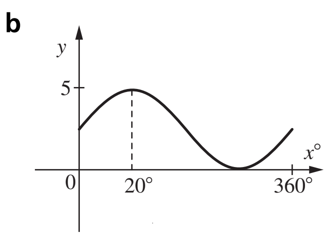
e $y = \arcsin\theta$

10 State the equation of the graph.

a



b



11 Solve these for $0 \le \theta < 2\pi$.

a $2\sin\theta - 1 = 0$ **b** $2\cos\theta + \sqrt{3} = 0$

c $6\tan\theta - 6 = 0$ **d** $2\sin 4\theta - \sqrt{3} = 0$

e $\sqrt{3}\tan 2\theta + 1 = 0$

12 Solve these for $0° \le x° < 360°$.

a $8\tan x° + 8 = 0$ **b** $9\sin x° = 9$

c $4\sin x° + 2 = 0$ **d** $\sqrt{3}\tan x° + 1 = 4$

e $6\cos 2x° = 3\sqrt{3}$ **f** $8\sin 3x° - 4 = 0$

13 Solve these for $0° \le x° < 360°$.

a $7\cos x° - 3 = 0$ **b** $8\sin 2x° + 5 = 0$

c $9\tan 3x° - 17 = 0$ **d** $3\sec x° - 7 = 0$

14 Solve $2\sin x = \tan x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. [IB May 01 P1 Q2]

15 The angle θ satisfies the equation $\tan\theta + \cot\theta = 3$ where θ is in degrees. Find all the possible values of θ lying in the interval [0°, 90°]. [IB May 02 P1 Q10]

16 The height in cm of a cylindrical piston above its central position is given by

$$h = 16\sin 4t$$

where t is the time in seconds, $0 \le t \le \frac{\pi}{4}$.

a What is the height after $\frac{1}{2}$ second? `

b Find the first time at which the height is 10 cm.

17 Let $f(x) = \sin\left(\arcsin\frac{x}{4} - \arccos\frac{3}{5}\right)$ for $-4 \le x \le 4$.

a Sketch the graph of $f(x)$.

b On the sketch, clearly indicate the coordinates of the x-intercept, the y-intercept, the minimum point and the endpoints of the curve of $f(x)$.

c Solve $f(x) = -\frac{1}{2}$. [IB Nov 03 P1 Q14]